Lie Theory Through Examples, J. Baez, Winter 2009.

The C_n Series

These are the compact simply-connected simple Lie groups:

$$Sp(n) = \{ n \times n \text{ quaternionic matrices } T \text{ w/ } TT^* = 1 \}$$

where T^* is the conjugate transpose of T:

$$(T^*)_{ij} = (T_{ji}^*)$$

where

$$(a + bi + cj + dk)^* = a - bi - cj - dk$$

We have

$$\begin{aligned} Sp(1) &= \{ \text{unit quaternions} \} \\ &\cong Spin(3) \end{aligned}$$

More generally $Sp(n)$ has maximal torus

$$T = \left\{ \begin{pmatrix} e^{j\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{j\theta_n} \end{pmatrix} : \theta_i \in \mathbb{R} \right\}$$

where

$$e^{j\theta} = \sum_{n=1}^{\infty} \frac{(j\theta)^n}{n!} = \cos\theta + j\sin\theta$$

So $\dim T = n$.

This gives

$$\mathfrak{t} = \left\{ \begin{pmatrix} j\theta_1 & & 0 \\ & \ddots & \\ 0 & & j\theta_n \end{pmatrix} : \theta_i \in \mathbb{R} \right\} \cong \mathbb{R}^n$$

& so

$$L = \{ x \in \mathfrak{t} : \exp(2\pi x) = 1 \}$$

$$= \left\{ \begin{pmatrix} j\theta_1 & & 0 \\ & \ddots & \\ 0 & & j\theta_n \end{pmatrix} : \theta_i \in \mathbb{Z} \right\} \cong \mathbb{Z}^n$$

So our lattice is just a "hypercubical" lattice, just like B_n.

For B_n, the Weyl group was $S_n \ltimes \mathbb{Z}_2^n$

How about C_n?

Find g with:

$$g \begin{pmatrix} e^{j\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{j\theta_n} \end{pmatrix} g^{-1} \in T$$

If g is a permutation matrix this holds, $W \supseteq S_n$. What else?

For example:

$$\begin{pmatrix} K & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{j\theta_1} & 0 \\ 0 & e^{j\theta_2} \end{pmatrix}\begin{pmatrix} K^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

$Kj = -jK$

$KjK^{-1} = -j$

$Ke^{j\theta}K^{-1} = e^{-j\theta}$

So we get

$$\begin{pmatrix} e^{-j\theta_1} & 0 \\ 0 & e^{j\theta_2} \end{pmatrix}$$

So, generalizing from this, we get $W \cong S_n \ltimes \mathbb{Z}_2^n$

In fact, that's all! $W = S_n \ltimes \mathbb{Z}_2^n$.

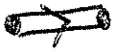
So B_n & C_n have the same Weyl group.

In what way do they differ???

We'll leave this ~~as~~ a puzzle for now.

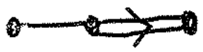
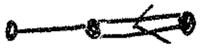
$$\mathfrak{so}(3) \cong B_1 = \bullet = C_1 \cong \mathfrak{sp}(1)$$

since we've seen $Sp(1)$ is the double cover of $SO(3)$.

$\mathfrak{so}(5) \cong B_2$  $\cong$  $C_2 \cong \mathfrak{sp}(2)$

Indeed, $Sp(2)$ is the double cover of $SO(5)$.

But:

$\mathfrak{so}(7) \cong B_3$  $\not\cong$  $C_3 \cong \mathfrak{sp}(3)$

Let's digress & consider all the coincidences between A_n, B_n, C_n, D_n's:

 $A_1 \cong B_1 \cong C_1$ (there's no D_1)

$\mathfrak{su}(2)$ $\mathfrak{so}(3)$ $\mathfrak{sp}(1)$

We know $Sp(1)$ double covers $SO(3)$, but also

$$\begin{tikzcd}
\mathbb{H} \arrow[r, hook] & \mathbb{C}\otimes_{\mathbb{R}}\mathbb{H} \cong M_2(\mathbb{C}) \\
Sp(1) \arrow[u, hook] \arrow[r, "\sim"] & SU(2) \arrow[u, hook]
\end{tikzcd}$$

gives $Sp(1) \cong SU(2)$, so $\mathfrak{su}(2) \cong \mathfrak{so}(3) \cong \mathfrak{sp}(1)$.

$A_1 \times A_1$

D_2

$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \cong \mathfrak{so}(4)$

$$SU(2) \times SU(2) \cong Spin(4)$$

Start with $\mathbb{R}^4$ w/ usual inner product & orientation. It's symmetry group is $SO(4)$. This acts on the exterior algebra $\Lambda \mathbb{R}^4$, preserving the Hodge star operator:

$$* : \Lambda^p \mathbb{R}^4 \to \Lambda^{4-p} \mathbb{R}^4$$

defined by

$$\omega \wedge * \nu = \langle \omega, \nu \rangle \, \mathrm{vol}, \quad \omega, \nu \in \Lambda^p \mathbb{R}^4$$

where $\langle \cdot, \cdot \rangle$ on $\Lambda \mathbb{R}^4$ is induced from the inner product on $\mathbb{R}^4$ & $\mathrm{vol} = e_1 \wedge e_2 \wedge e_3 \wedge e_4 \in \Lambda^4 \mathbb{R}^4$, where $e_1, \ldots, e_4$ is any oriented basis.

We have

$$* : \Lambda^2 \mathbb{R}^4 \longrightarrow \Lambda^2 \mathbb{R}^4$$

and

$$*(e_i \wedge e_j) = e_k \wedge e_l,$$ where i, j, k, l are an even permutation of 1, 2, 3, 4.

$$** (e_1 \wedge e_2) = *(e_3 \wedge e_4) = e_1 \wedge e_2$$

& in general $*^2 = 1$ on $\Lambda^2 \mathbb{R}^4$. So

$$\Lambda^2 \mathbb{R}^4 \cong \Lambda^2_+ \mathbb{R}^4 \oplus \Lambda^2_- \mathbb{R}^4$$

Where $*\mu = \pm \mu$ if $\mu \in \Lambda^2_\pm \mathbb{R}^4$.

So: $SO(4)$ acts on $\Lambda^2 \mathbb{R}^4$ preserving this splitting & inner product. So we get a homomorphism:

$$SO(4) \longrightarrow SO(\Lambda^2_+ \mathbb{R}^4) \times SO(\Lambda^2_- \mathbb{R}^4)$$

$$\dim \Lambda^2 \mathbb{R}^4 = \binom{4}{2} = 6$$

$$\Lambda^2_+ \mathbb{R}^4 = \langle e_1\wedge e_2 + e_3\wedge e_4,\ e_1\wedge e_3 + e_4\wedge e_2,\ e_1\wedge e_4 + e_2\wedge e_3 \rangle$$

has dim = 3, & similarly for $\Lambda^2_- \mathbb{R}^3$.

So we get:

$$SO(4) \longrightarrow SO(3) \times SO(3)$$

This is 2-1, so

$$\mathfrak{so}(4) \longrightarrow \mathfrak{so}(3) \oplus \mathfrak{so}(3)$$

is 1-1. Since both sides have dimension 6, it's onto. Thus

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$$

So

$$\mathrm{Spin}(4) \cong \mathrm{Spin}(3) \times \mathrm{Spin}(3)$$
$$\cong SU(2) \times SU(2)$$

Next:

 $\cong$ 

A_3 D_3

$$\mathfrak{sl}(4) \cong \mathfrak{so}(6)$$

$$SU(4) \cong \mathrm{Spin}(6)$$

$SU(4)$ is the symmetries of $\mathbb{C}^4$ w/ its usual inner product & volume form $e_1 \wedge e_2 \wedge e_3 \wedge e_4 \in \Lambda^2 \mathbb{C}^4$.

We can define a Hodge star operator
$$* \; \Lambda^2 \mathbb{C}^4 \longrightarrow \Lambda^2 \mathbb{C}^4$$

w/ $*^2 = 1$ as before, but now $*$ is conjugate linear, since

$$\omega \wedge * \nu = \langle \underset{\text{linear}}{\omega}, \underset{\text{conjugate linear}}{\nu} \rangle \, \mathrm{vol}$$

Now

$$\Lambda^2_\pm \mathbb{C}^4 = \{ * \omega = \pm \omega \}$$

are <u>real</u> subspaces of $\Lambda^2 \mathbb{C}^4$, w/ real dimension 6.

So we get 2-1 map

$$SU(4) \longrightarrow SO(\Lambda^2_+ \mathbb{C}^4) \cong SO(6)$$

So

$$\Delta U(4) \longrightarrow \Delta O(6)$$

is 1-1, & thus onto since

$$\dim SU(4) = 4^2 - 1 = 15$$

$$\dim SO(6) = \frac{6(6-1)}{2} = 15$$

& thus the dimensions are equal.

Thus

$$\mathfrak{su}(4) \cong \mathfrak{so}(6)$$

&

$$SU(4) \cong Spin(6).$$

The last coincidence will be

$$B_2 \quad \circ\!\Rightarrow\!\circ \quad \cong \quad \circ\!\Leftarrow\!\circ \quad C_2$$

$$\mathfrak{so}(5) \cong \mathfrak{sp}(2)$$

$$Spin(5) \cong Sp(2)$$

to be discussed next time!