Lie Theory Through Examples

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A_n

$$K = SU(n+1)$$

$$T = \left\{ \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_{n+1}} \end{pmatrix} : \sum \theta_i = 0 \right\}$$

$$\mathfrak{t} = \left\{ \begin{pmatrix} i\theta_1 & & \\ & \ddots & \\ & & i\theta_{n+1} \end{pmatrix} : \sum \theta_i = 0 \right\}$$

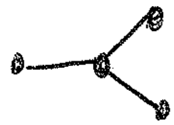
$$\begin{array}{c} \cup \\ L \end{array} \cong \left\{ (x_1, \ldots, x_{n+1}) : x_i \in \mathbb{Z}, \sum x_i = 0 \right\}$$

$$\theta_i = 2\pi x_i.$$

$$W = S_{n+1}$$

$W \sim$ permutations of coordinates.

D_n



$$K = \widetilde{SO}(2n) = \mathrm{Spin}(2n)$$

$$T = \left\{ \begin{pmatrix} e^{J\theta_1} & & \\ & \ddots & \\ & & e^{J\theta_n} \end{pmatrix} \right\}$$ double cover

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\mathfrak{t} = \left\{ \begin{pmatrix} J\theta_1 & & \\ & \ddots & \\ & & J\theta_n \end{pmatrix} \right\}$$

$$\begin{array}{c} \cup \\ L \end{array} \cong \left\{ (x_1, \ldots, x_n) : x_i \in \mathbb{Z}, \sum x_i \in 2\mathbb{Z} \right\}$$

$$W = S_n \ltimes \mathbb{Z}_2^{n-1}$$

$W \sim$ permutation of coordinates & even numbers of sign changes.

B_n

$K = \widetilde{SO}(2n+1) = Spin(2n+1)$

$$T = \left\{ \begin{pmatrix} e^{J\theta_1} & & & 0 \\ & \ddots & & \\ & & e^{J\theta_n} & \\ 0 & & & 1 \end{pmatrix} \right\}_{\text{double cover}}$$

$$\mathfrak{t} = \left\{ \begin{pmatrix} J\theta_1 & & & 0 \\ & \ddots & & \\ & & J\theta_n & \\ 0 & & & 0 \end{pmatrix} \right\}$$

$\rotatebox{90}{$\subseteq$}$

$L \cong \{(x_1, \dots, x_n) : x_i \in \mathbb{Z}, \sum x_i \in 2\mathbb{Z}\}$

$W = S_n \ltimes \mathbb{Z}_2^n$

$W \sim$ permutations & sign changes of coordinates.

C_n

$K = Sp(n)$

$$T = \left\{ \begin{pmatrix} e^{j\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{j\theta_n} \end{pmatrix} \right\}, \quad j \in \mathbb{H}$$

$$\mathfrak{t} = \left\{ \begin{pmatrix} j\theta_1 & & 0 \\ & \ddots & \\ 0 & & j\theta_n \end{pmatrix} \right\}$$

$\rotatebox{90}{$\subseteq$}$

$L \cong \{(x_1, \dots, x_n) : x_i \in \mathbb{Z}\}$

$W = S_n \ltimes \mathbb{Z}_2^n$

$W \sim$ permutations & sign changes of coordinates.

ROOTS

Recall any complex rep of K restricts to a rep of the maximal torus $T \subseteq K$. Any rep of T is a direct sum of 1-dim irreps, one for each point

$$\ell \in L^* = \{\ell : \mathfrak{t} \to \mathbb{R} : \ell(x) \in \mathbb{Z} \ \forall x \in L\}$$

Namely:

$$\rho_\ell(\exp(x)) = e^{2\pi i \ell(x)} \in U(1)$$
$$x \in \mathfrak{t}$$

So given any complex rep ρ of K we get a function

$$d_\rho : L^* \to \mathbb{N}$$

where $d_\rho(\ell)$ says how many times ρ_ℓ shows up in the direct sum decomposition of ρ.

K always has a rep on its own Lie algebra, $\mathfrak{k}$, the adjoint rep.

This is a real rep, but $\mathbb{C} \otimes \mathfrak{k}$ is a complex rep, so we can play this game, & get a weighting canonically associated to K.

<u>Example</u>:

$$K = SU(3)$$
$$\mathfrak{k} = \mathfrak{su}(3)$$
$$\mathfrak{g} = \mathbb{C} \otimes \mathfrak{k} = \mathfrak{sl}(3, \mathbb{C})$$

$$G = SL(3, \mathbb{C})$$

The rep of $T \subseteq K$ on $\mathfrak{g} = \mathbb{C} \otimes \mathfrak{k}$

is:

$$\begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix}^{-1}$$

$$a + e + i = 0$$

$$= \begin{pmatrix} e^{i(\theta_1 - \theta_1)} a & e^{i(\theta_1 - \theta_2)} b & e^{i(\theta_1 - \theta_3)} c \\ e^{i(\theta_2 - \theta_1)} d & e^{i(\theta_2 - \theta_2)} e & e^{i(\theta_2 - \theta_3)} f \\ e^{i(\theta_3 - \theta_1)} g & e^{i(\theta_3 - \theta_2)} h & e^{i(\theta_3 - \theta_3)} i \end{pmatrix}$$

So we see $\mathfrak{g}$ splits as a sum of 8 irreps, 6 off-diagonal ones like

$$\begin{pmatrix} 0 & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & e^{i(\theta_1 - \theta_2)} b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

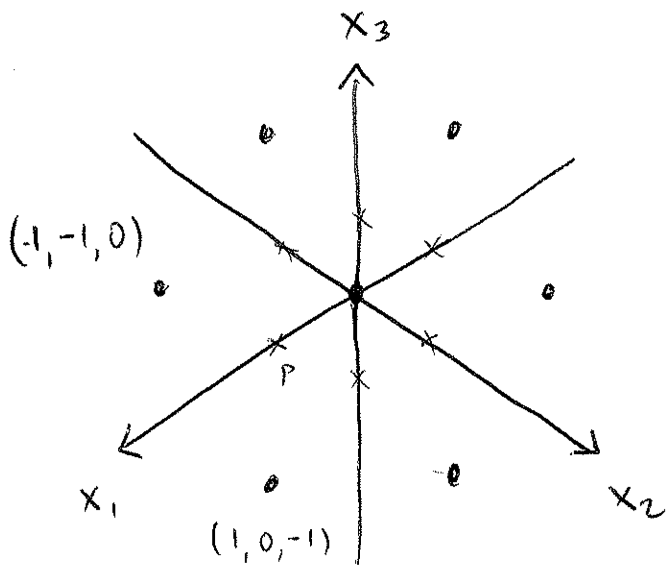
& 2 diagonal ones like

$$\begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 0 \end{pmatrix} \longmapsto \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & -e \end{pmatrix} \longmapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & -e \end{pmatrix}$$

are trivial reps.

In general we get n trivial reps if T is n-dimensional, from $\mathbb{C} \otimes \mathfrak{t} \subseteq \mathbb{C} \otimes \mathfrak{k} = \mathfrak{g}$.



$\mathfrak{t} = \{(\theta_1, \theta_2, \theta_3) : \sum \theta_i = 0\}$

$L = \{(x_1, x_2, x_3) : \sum x_i = 0, \times$

$\theta_i = 2\pi x_i$

$L^* \subseteq \mathfrak{t}^*$ but we can pretend $L^* \subseteq \mathfrak{t}$ using $\mathfrak{t} \cong \mathfrak{t}^*$ coming from the inner product on $\mathfrak{t}$.

Using this we get $L^* \subseteq L$, but we get more points than in L alone.

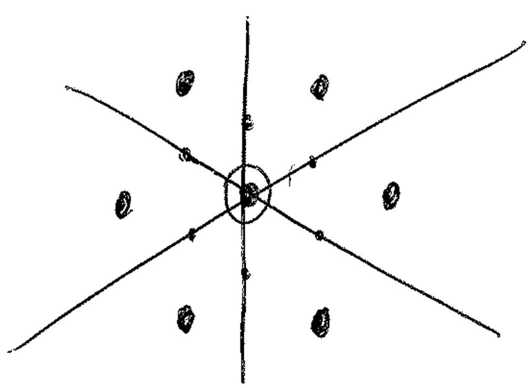
For example

$$p = \frac{1}{3}\left((0,0,0) + (1,0,-1) + (1,-1,0) \right)$$

$$= \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$$

is in L^* but not L!

The 8 irreps of T sitting in the rep $\mathbb{C} \otimes k \cong \mathfrak{sl}(3, \mathbb{C})$ correspond to these 8 pts in L^*:



L^*

The nonzero points in this picture are roots. For any K there will be $\dim K - \dim T$ roots: here $6 = 8-2$.

In fact the roots always lie in L, not just L^*.