Algebraic structures from diagrams

Julie Bergner

University of California, Riverside

September 17, 2011

Algebraic theories

Definition

An *algebraic theory* $\mathcal{T}$ is a category with objects T_n, $n \in \mathbb{N}$ such that $T_n \cong (T_1)^n$ for all n. In particular, T_0 is a terminal object of $\mathcal{T}$.

Algebraic theories

Definition

An *algebraic theory* $\mathcal{T}$ is a category with objects T_n, $n \in \mathbb{N}$ such that $T_n \cong (T_1)^n$ for all n. In particular, T_0 is a terminal object of $\mathcal{T}$.

Example

Take the full subcategory of finitely generated free groups in the category of groups.

If T_n is the free group on n generators, then $T_n \cong \underbrace{T_1 * \cdots * T_1}_{n}$.

Taking the opposite category gives $T_n \cong \underbrace{T_1 \times, \cdots \times T_1}_{n}$.

Call this category the *theory of groups*, denoted $\mathcal{T}_G$.

Why theories?

Theorem (Lawvere)

The category of product-preserving functors $\mathcal{T}_G \to \mathcal{S}ets$ is equivalent to the category of groups.

Why theories?

Theorem (Lawvere)

The category of product-preserving functors $\mathcal{T}_G \to \mathcal{S}ets$ is equivalent to the category of groups.

More generally, we can find an associated theory to any category of algebraic objects for which free objects are defined.

Example

Taking free monoids rather than free groups in the previous example gives the theory of monoids, $\mathcal{T}_M$.

Definition

Let $\mathcal{T}$ be an algebraic theory. A *strict* $\mathcal{T}$*-algebra* is a product-preserving functor $A\colon \mathcal{T} \to s\mathcal{S}ets$. In other words, the projection functors $p_n\colon T_n \to T_1$ induce an isomorphism of simplicial sets $A(T_n) \cong A(T_1)^n$ for each n.

Example

Strict $\mathcal{T}_G$-algebras are equivalent to simplicial (or topological) groups.

Theories in homotopy theory

Definition

Let $\mathcal{T}$ be an algebraic theory. A *strict* $\mathcal{T}$*-algebra* is a product-preserving functor $A \colon \mathcal{T} \to s\mathcal{S}ets$. In other words, the projection functors $p_n \colon T_n \to T_1$ induce an isomorphism of simplicial sets $A(T_n) \cong A(T_1)^n$ for each n.

Example

Strict $\mathcal{T}_G$-algebras are equivalent to simplicial (or topological) groups.

Definition

A *homotopy* $\mathcal{T}$*-algebra* is a functor $B \colon \mathcal{T} \to s\mathcal{S}ets$ such that the projection functors induce a weak equivalence $B(T_n) \simeq B(T_1)^n$ for all n.

How weak are homotopy algebras?

Let $\mathcal{A}lg^{\mathcal{T}}$ denote the category of strict $\mathcal{T}$-algebras.

There is a model structure on $\mathcal{A}lg^{\mathcal{T}}$ with weak equivalences and fibrations given by evaluation at T_1.

There is no model structure on the category of homotopy $\mathcal{T}$-algebras, but there is a model structure $\mathcal{L}s\mathcal{S}ets^{\mathcal{T}}$ on the category of functors $\mathcal{T} \to s\mathcal{S}ets$ in which the fibrant objects are homotopy $\mathcal{T}$-algebras.

How weak are homotopy algebras?

Let $\mathcal{A}lg^{\mathcal{T}}$ denote the category of strict $\mathcal{T}$-algebras.

There is a model structure on $\mathcal{A}lg^{\mathcal{T}}$ with weak equivalences and fibrations given by evaluation at T_1.

There is no model structure on the category of homotopy $\mathcal{T}$-algebras, but there is a model structure $\mathcal{L}s\mathcal{S}ets^{\mathcal{T}}$ on the category of functors $\mathcal{T} \to s\mathcal{S}ets$ in which the fibrant objects are homotopy $\mathcal{T}$-algebras.

Theorem (Badzioch)

There is a Quillen equivalence between $\mathcal{A}lg^{\mathcal{T}}$ and $\mathcal{L}s\mathcal{S}ets^{\mathcal{T}}$.

Are there simpler diagrams?

In general, algebraic theories can have a lot of morphisms.

For example, in $\mathcal{T}_M$, the maps $T_2 \rightarrow T_1$ are given by monoid homomorphisms $T_1 \rightarrow T_2$. They in turn correspond to finite words on two letters.

Are there simpler diagrams?

In general, algebraic theories can have a lot of morphisms.

For example, in $\mathcal{T}_M$, the maps $T_2 \to T_1$ are given by monoid homomorphisms $T_1 \to T_2$. They in turn correspond to finite words on two letters.

One might guess that a simpler category might encode the structure of a monoid just as well.

How could we find one?

Segal monoids

Definition

A *Segal monoid* is a functor $X \colon \mathbf{\Delta}^{op} \to s\mathcal{S}ets$ such that $X_0 = \Delta[0]$ and the Segal maps $X_n \to (X_1)^n$ are weak equivalences for $n \geq 2$.

There is a model structure $\mathcal{L}s\mathcal{S}ets_*^{\mathbf{\Delta}^{op}}$ on the category of functors $X \colon \mathbf{\Delta}^{op} \to s\mathcal{S}ets$ with $X_0 = \Delta[0]$, with fibrant objects Segal monoids.

Segal monoids

Definition

A *Segal monoid* is a functor $X \colon \mathbf{\Delta}^{op} \to s\mathcal{S}ets$ such that $X_0 = \Delta[0]$ and the Segal maps $X_n \to (X_1)^n$ are weak equivalences for $n \geq 2$.

There is a model structure $\mathcal{L}s\mathcal{S}ets_*^{\mathbf{\Delta}^{op}}$ on the category of functors $X \colon \mathbf{\Delta}^{op} \to s\mathcal{S}ets$ with $X_0 = \Delta[0]$, with fibrant objects Segal monoids.

Theorem (B)

There is a Quillen equivalence between $\mathcal{L}s\mathcal{S}ets_*^{\mathbf{\Delta}^{op}}$ *and* $\mathcal{L}s\mathcal{S}ets_*^{\mathcal{T}_M}$.

Combining with Badzioch's result, we get that a Segal monoid can be rigidified to a simplicial monoid.

Multi-sorted theories

Definition

Given a set S, an *S-sorted algebraic theory* $\mathcal{T}$ is a small category with objects $T_{\underline{\alpha}^n}$ where $\underline{\alpha}^n =< \alpha_1, \ldots, \alpha_n >$ for $\alpha_i \in S$ and $n \geq 0$ varying, and such that

$$T_{\underline{\alpha}^n} \cong \prod_{i=1}^{n} T_{\alpha_i}.$$

There exists a terminal object T_0 (corresponding to the empty subset of S).

For a multi-sorted theory $\mathcal{T}$, we can still define strict and homotopy algebras, and Badzioch's theorem still holds for their respective model structures.

The theory of operads

Multi-sorted theories can encode more complicated algebraic structures where free objects still make sense.

Example

Consider the category of finitely generated free operads. The opposite of this category is $\mathcal{T}_{\mathcal{O}}$, the *theory of operads*. The “sorts” in this theory are the natural numbers, since a generator can occur in any arity.

Can we find a simpler diagram to encode the structure of an operad?

The category Ω

Definition (Moerdijk-Weiss)

Let Ω be the category whose objects are rooted trees and whose morphisms are given by the operad morphisms between the (colored) operads generated by them. Denote by η the tree with no vertices and one edge.

Definition

A *dendroidal set* is a functor $\Omega^{op} \to \mathcal{S}ets$.

Given an object T of Ω, let $\Omega[T]$ denote the representable functor $\mathrm{Hom}_{\Omega}(-, T)$, called the *standard T-dendrex*.

Dendroidal spaces

Similarly, a *dendroidal space* is a functor $X\colon \Omega^{op} \to s\mathcal{S}ets$.

Consider the category of dendroidal spaces X such that $X(\eta) = \Delta[0]$.

We can also impose a condition similar to the Segal condition for simplicial spaces.

Dendroidal spaces

Similarly, a *dendroidal space* is a functor $X\colon \Omega^{op} \to s\mathcal{S}ets$.

Consider the category of dendroidal spaces X such that $X(\eta) = \Delta[0]$.

We can also impose a condition similar to the Segal condition for simplicial spaces.

Definition

Let T be an object of Ω. Its *Segal core* $Sc[T]$ is the subobject of the T-dendrex given by all the corollas in T. In other words,

$$Sc[T] = \bigcup_v \Omega[C_{n(v)}]$$

where v denotes a vertex of T, $n(v)$ denotes the number of input vertices of v, and C_n denotes the n-corolla.

So we have an inclusion $Sc[T] \to \Omega[T]$ for any tree T which we can use to define Segal maps.

Definition

A *Segal operad* is a dendroidal space $X \colon \Omega^{op} \to s\mathcal{S}ets$ such that $X(\eta) = \Delta[0]$ and the induced map

$$\mathrm{Map}(\Omega[T], X) \to \mathrm{Map}(Sc[T], X)$$

is a weak equivalence of simplicial sets for all objects T of Ω.

Since we have imposed the condition that $X(\eta) = \Delta[0]$, we expect that Segal operads should correspond to ordinary (single-colored) operads.

Rigidification of Segal operads

Theorem (B-Hackney)

There is a model structure $\mathcal{L}s\mathcal{S}ets_*^{\Omega^{op}}$ on the category of dendroidal spaces X with $X(\eta) = \Delta[0]$ such that the fibrant objects are Segal operads.

Rigidification of Segal operads

Theorem (B-Hackney)

There is a model structure $\mathcal{L}s\mathcal{S}ets_*^{\Omega^{op}}$ on the category of dendroidal spaces X with $X(\eta) = \Delta[0]$ such that the fibrant objects are Segal operads.

Theorem (B-Hackney)

The model structure $\mathcal{L}s\mathcal{S}ets_*^{\Omega^{op}}$ is Quillen equivalent to the model structure $\mathcal{L}s\mathcal{S}ets^{\mathcal{T}_\mathcal{O}}$.

In other words, together with the multi-sorted version of Badzioch's theorem, we get that any Segal operad can be rigidified to a simplicial operad.

Generalizations

The rigidification result for Segal monoids can be generalized to one for more general Segal categories with a fixed set of objects, still using algebraic theories.

The more general result, that the model categories for Segal categories and simplicial categories are Quillen equivalent, is no longer formulated in terms of algebraic theories.

Generalizations

The rigidification result for Segal monoids can be generalized to one for more general Segal categories with a fixed set of objects, still using algebraic theories.

The more general result, that the model categories for Segal categories and simplicial categories are Quillen equivalent, is no longer formulated in terms of algebraic theories.

It is expected that we can prove an analogous result for operads with a fixed set of colors.

The Quillen equivalence between colored Segal operads and simplicial colored operads (without fixing the color) has been established by Cisinski and Moerdijk.

Further work

Group actions on algebraic objects can be encoded by (multi-sorted) algebraic theories in two ways.

First, given an algebraic theory $\mathcal{T}$ and a fixed group G, there is a theory whose algebras are algebras over $\mathcal{T}$ equipped with a G-action.

Further work

Group actions on algebraic objects can be encoded by (multi-sorted) algebraic theories in two ways.

First, given an algebraic theory $\mathcal{T}$ and a fixed group G, there is a theory whose algebras are algebras over $\mathcal{T}$ equipped with a G-action.

Second, given an algebraic theory $\mathcal{T}$, there is a theory whose algebras are algebras over $\mathcal{T}$ with a group action by some group G.

Further work

Group actions on algebraic objects can be encoded by (multi-sorted) algebraic theories in two ways.

First, given an algebraic theory $\mathcal{T}$ and a fixed group G, there is a theory whose algebras are algebras over $\mathcal{T}$ equipped with a G-action.

Second, given an algebraic theory $\mathcal{T}$, there is a theory whose algebras are algebras over $\mathcal{T}$ with a group action by some group G.

For both monoids and operads, we expect that we can translate each of these constructions to a Segal-type setting.