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Complete Segal spaces as a model for simplicial categories

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Simplicial Categories

Definition 1. A (small) *simplicial category* is a category with a set of objects and a simplicial set of morphisms between any two objects.

We will denote the category of simplicial categories $\mathcal{SC}$.

Why do we care about simplicial categories?

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Recall: Given a model category $\mathcal{M}$, there is associated to it $\mathrm{Ho}(\mathcal{M})$, its homotopy category.

There is also its *simplicial localization* $\mathcal{LM}$ which is a simplicial category encoding all the homotopy-theoretic information about $\mathcal{M}$.

Hence, we can consider the study of simplicial categories to be the study of homotopy theories.

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First, we need a notion of "weak equivalence" of simplicial categories.

Definition 2. A map $f : \mathcal{C} \to \mathcal{D}$ of simplicial categories is a *DK-equivalence* if:

- $\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(fx, fy)$ is a weak equivalence of simplicial sets for any objects x, y of $\mathcal{C}$.
- $\pi_0\mathcal{C} \to \pi_0\mathcal{D}$ is an equivalence of categories

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The problem is that simplicial categories (and their weak equivalences) are difficult to work with, so we would like to find another, nicer category in which to obtain information about homotopy theories.

We will try to use complete Segal spaces instead.

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Definition 3. A *Segal space* is a Reedy fibrant simplicial space W such that the Segal maps

$$\varphi_k : W_k \to \underbrace{W_1 \times_{W_0} \cdots \times_{W_0} W_1}_{k}$$

are weak equivalences of simplicial sets for $k \geq 2$.

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We can talk about Segal spaces much like we talk about (simplicial) categories:

- "objects": the set $W_{0,0}$
- "morphism space" $\mathrm{map}_W(x, y) =$ the fiber over (x, y) of the map $W_1 \to W_0 \times W_0$

Can also define "compositions" and "homotopy equivalences."

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There is a space of homotopy equivalences

$$W_{\mathrm{hoequiv}} \subseteq W_1.$$

Note that the degeneracy map $s_0 : W_0 \to W_1$ factors through W_{hoequiv}:

$$\begin{tikzcd} W_0 \arrow[rr, "s_0"] \arrow[dr] & & W_1 \\ & W_{\mathrm{hoequiv}} \arrow[ur] & \end{tikzcd}$$

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We are now able to define the objects that we want to work with.

Definition 4. A *complete Segal space* W is a Segal space such that the map $W_0 \to W_{\text{hoequiv}}$ is a weak equivalence of simplicial sets.

We will denote the category of complete Segal spaces by $\mathcal{CSS}$.

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Going from Simplicial Categories to Complete Segal Spaces

In the case where $\mathcal{C}$ is discrete (not simplicial), there is a functor from $\mathcal{SC}$ to $\mathcal{CSS}$ which is easy to describe.

Definition 5. Let $\mathcal{C}$ be a category. Then its *classifying diagram* $N\mathcal{C}$ is the simplicial space defined by

$$(N\mathcal{C})_n = \text{nerve}(\text{iso } \mathcal{C}^{[n]}).$$

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So, for example, $(N\mathcal{C})_0$ is the nerve of the maximal subgroupoid of $\mathcal{C}$.

Given any category $\mathcal{C}$, $N\mathcal{C}$ is a complete Segal space.

Example 6. Suppose G is a group. Then up to homotopy NG looks like the constant simplicial space

$$BG \Leftarrow BG \Lleftarrow BG \cdots$$

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Example 7. If $\mathcal{C}$ is a groupoid, then up to homotopy $N\mathcal{C}$ looks like the constant simplicial space

$$\coprod_{<x>} BAut(x) \Leftarrow \coprod_{<x>} BAut(x) \Lleftarrow \cdots$$

where $< x >$ denotes the isomorphism classes of objects of $\mathcal{C}$.

In general, given any simplicial category $\mathcal{C}$, its complete Segal space cannot be computed so easily, since the $N\mathcal{C}$ construction is no longer homotopy invariant.

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Before we define our functor in this case, recall that the nerve of a simplicial category $\mathcal{C}$ is a simplicial space which looks like

$$Ob(\mathcal{C}) \Leftarrow Mor(\mathcal{C}) \Lleftarrow Mor(\mathcal{C}) \times_{Ob(\mathcal{C})} Mor(\mathcal{C}) \cdots$$

Definition 8. A *Segal category* X is a simplicial space such that:

- X_0 is discrete, and
- the Segal map $\varphi_k : X_k \to \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{k}$ is a weak equivalence of simplicial sets for $k \geq 2$.

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Note that the nerve of a simplicial category is a Segal category.

We will denote the category of Segal categories by $\mathcal{S}e\mathcal{C}at$.

Rezk defines a composite functor

$$\mathcal{SC} \xrightarrow{\text{nerve}} \mathcal{S}e\mathcal{C}at \xrightarrow{\text{localization}} \mathcal{CSS}$$

The idea behind the "localization" functor is similar to the $N\mathcal{C}$ construction.

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Given any simplicial category $\mathcal{C}$ the corresponding complete Segal space up to homotopy looks like

$$\coprod_{<x>} BAut^h(x) \Leftarrow \coprod_{<x>,<y>} BAut^h(\coprod_{<\alpha>} \mathrm{Hom}(x,y)_\alpha) \Lleftarrow \cdots$$

where Aut^h denotes homotopy automorphisms and the $< \alpha >$ index the isomorphism classes of maps.

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Is There an Inverse Functor?

Up to DK-equivalence of simplicial categories, yes.

We first define a functor

$$\mathcal{CSS} \to \mathcal{S}e\mathcal{C}at.$$

The idea is to "discretize" the degree zero space but make sure that the Segal maps are still weak equivalences.

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Given a complete Segal space W, take the pullback at each level of the diagram of simplicial spaces:

$$\begin{tikzcd}
\vdots & \vdots & \vdots \\
W_2 \arrow[r] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & W_0 \times W_0 \times W_0 \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & W_{0,0} \times W_{0,0} \times W_{0,0} \arrow[l] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\
W_1 \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & W_0 \times W_0 \arrow[d, shift left] \arrow[d, shift right] & W_{0,0} \times W_{0,0} \arrow[l] \arrow[d, shift left] \arrow[d, shift right] \\
W_0 \arrow[r] & W_0 & W_{0,0} \arrow[l]
\end{tikzcd}$$

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The resulting simplicial space will actually be a Segal category.

We now apply a functor $\mathcal{S}e\mathcal{C}at \to \mathcal{SC}$ which “rigidifies” a Segal category.

Namely, in order to have a simplicial category, we need the map

$$X_k \to \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{k}$$

to be an isomorphism of simplicial sets, rather than a weak equivalence.

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We can apply a generalization of a theorem of Badzioch to obtain such a functor.

So we can now take a composite functor

$$\mathcal{SC} \xleftarrow{\text{rigidify}} \mathcal{S}e\mathcal{C}at \xleftarrow{\text{discretize}} \mathcal{CSS}$$

Theorem 9. *If we apply Rezk's functor to a simplicial category $\mathcal{C}$, followed by this composite, the simplicial category we get is DK-equivalent to $\mathcal{C}$.*

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Model Categories

There is a model category structure on the category of simplicial categories such that the weak equivalences are the DK-equivalences.

There is also a model category structure on the category of simplicial spaces such that the fibrant-and-cofibrant objects are the complete Segal spaces.

Conjecture 10. *These two model category structures are Quillen equivalent.*

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References

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