Homotopical versions of Hall algebras

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January 7, 2009

Classical Hall algebras

Let $\mathcal{A}$ be an abelian category such that, for any objects X and Y of $\mathcal{A}$, $Ext^1(X, Y)$ is finite.

Then, we can associate to $\mathcal{A}$ an associative algebra $\mathcal{H}(\mathcal{A})$ called the *Hall algebra*.

As a vector space, $\mathcal{H}(\mathcal{A})$ has as basis the isomorphism classes of objects of $\mathcal{A}$.

Then, given isomorphism classes X and Y, we define their product to be given by

$$\sum_Z g_{X,Y}^Z Z$$

where the *Hall number* $g_{X,Y}^Z$ counts the number of short exact sequences

$$0 \to X \to Z \to Y \to 0$$

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For example, let A be a finite-dimensional algebra over $\mathbb{F}_q$, a finite field. The category of A-modules is abelian, so we can take $\mathcal{H}(A)$ as its Hall algebra.

Hall algebras associated to Lie algebras

Let $\mathfrak{g}$ be a Lie algebra of type A, D, or E. Recall that, $\mathfrak{g}$ is associated with a simply-laced Dynkin diagram, which is just an unoriented graph.

To this diagram, we can associate a quiver Q, where we assign a direction to each edge. (The following construction is independent of which directions we assign.)

Let $\mathbb{F}_q$ be a finite field. The category $Rep(Q)$ of $\mathbb{F}_q$-representations of Q forms an abelian category $\mathcal{A}$, and it has an associated Hall algebra $\mathcal{H}(\mathfrak{g})$.

Hall algebras and quantum groups

Also associated to $\mathfrak{g}$ is the quantum group $U_q(\mathfrak{g})$.

The Lie algebra $\mathfrak{g}$ can be decomposed as $\mathfrak{n}^+ \otimes \mathfrak{h} \otimes \mathfrak{n}^-$; letting $\mathfrak{b} = \mathfrak{n}^+ \otimes \mathfrak{h}$, we can consider the quantum group $U_q(\mathfrak{b})$.

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Question: Can we find a larger algebra associated to $\mathfrak{g}$ from which one could recover all of $U_q(\mathfrak{g})$?

Peng and Xiao conjectured that this larger algebra should be obtained from the derived category $\mathcal{D}(Rep(Q))$.

How can we construct an algebra from $\mathcal{D}(Rep(Q))$?

The ordinary Hall algebra construction does not work, since $\mathcal{D}(Rep(Q))$ is triangulated but not abelian; we do not necessarily have exact sequences.

One idea was to count distinguished triangles instead, but it was shown that the resulting algebra was not necessarily associative.

Toën has defined such a "derived Hall algebra" in some special cases of triangulated categories.

Definition

A *dg category* $\mathcal{T}$ is a category enriched over cochain complexes, so that for any objects x and y of $\mathcal{T}$. there is a cochain complex $\mathcal{T}(x,y)$.

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A dg category $\mathcal{T}$ is *locally finite* if for any objects x and y of $\mathcal{T}$, $\mathcal{T}(x, y)$ is cohomologically bounded and has finite-dimensional cohomology groups.

Let $\mathcal{M}(\mathcal{T})$ be the category of modules over $\mathcal{T}$, or dg functors from $\mathcal{T}$ to the category of cochain complexes over $\mathbb{F}_q$.

Consider the projective model structure on $\mathcal{M}(\mathcal{T})$, where weak equivalences and fibrations are given levelwise.

This model structure is stable, in that its homotopy category $Ho(\mathcal{M}(\mathcal{T}))$ is triangulated. We define

$$Ext^i(x, y) = [x, y[i]]$$

using the shift functor in this category.

One can restrict to the *perfect* objects of $\mathcal{M}(\mathcal{T})$, which are those objects still having nice finiteness properties, and from this subcategory $\mathcal{P}(\mathcal{T})$ one can define the derived Hall algebra $\mathcal{DH}(\mathcal{T})$.

Although we restrict to this subcategory, the original model structure plays an important role in the proof that $\mathcal{DH}(\mathcal{T})$ is associative.

Consider the category $w(\mathcal{P}(\mathcal{T})^{cof})$ of weak equivalences between objects of $\mathcal{M}(\mathcal{T})$ which are both perfect and cofibrant. Denote by $X(\mathcal{T})$ its nerve.

As a vector space, $\mathcal{DH}(\mathcal{T})$ is given by functions $X(\mathcal{T}) \to \mathbb{Q}$ with finite support.

Theorem (Toën)

The multiplication in $\mathcal{DH}(\mathcal{T})$ is given by

$$x \cdot y = \sum_z g_{x,y}^z z$$

where

$$g_{x,y}^z = \frac{|[x,z]_y| \cdot \prod_{i>0} |Ext^{-i}(x,z)|^{(-1)^i}}{|Aut(x)| \cdot \prod_{i>0} |Ext^{-i}(x,x)|^{(-1)^i}}.$$

Toën also proves that the algebra $\mathcal{DH}(\mathcal{T})$ only depends on the homotopy category $Ho(\mathcal{M}(\mathcal{T}))$. However, he uses the model structure on $\mathcal{M}(\mathcal{T})$ extensively in his proof.

It can be shown, using Toën's proof, that the specific model category $\mathcal{M}(\mathcal{T})$ is not necessary; one can generalize it to more general stable model categories satisfying certain finiteness assumptions.

Xiao and Xu also showed that Toën's formula does result in an associative algebra for any suitably finitary triangulated category, and their proof does not use model category techniques.

The problem is that $\mathcal{D}(Rep(Q))$ does not satisfy the necessary finiteness assumptions.

One idea is to find a way to build $\mathcal{D}(Rep(Q))$ as a colimit of finitary triangulated categories, so that we can use Toën's methods.

However, it is unlikely that we could get all the maps to be compatible with the model structures in each case, even if we could find the right model categories to use.

Therefore, it seems like a better option to use a more general notion of homotopy theory here, in particular one in which we have a model structure and can apply homotopy colimits.

Stable complete Segal spaces

Definition

A simplicial space W is a *Segal space* if the Segal maps

$$W_n \to W_1 \times_{W_0} \cdots \times_{W_0} W_1$$

are weak equivalences for all $n \geq 2$.

Since this map gives a notion of composition in a Segal space, at least up to homotopy, we can consider the objects of W, given by the set $W_{0,0}$, the mapping space $map(x, y)$ associated to any pair of objects x and y.

There is also a notion of homotopy equivalence, leading to the definition of the space of homotopy equivalences $W_h \subseteq W_1$, and a homotopy category $Ho(W)$.

Definition

A Segal space is *complete* if the degeneracy map $W_0 \to W_h$ is a weak equivalence. It is *stable* if the homotopy category is triangulated.

Derived Hall algebras for stable complete Segal spaces

Let W be a stable complete Segal space such that its homotopy category is finitary.

Then we can define the derived Hall algebra $\mathcal{DH}(W)$ corresponding to W.

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The hope is that this construction will be useful in the comparison with quantum groups.