Homotopy theory of higher categorical structures

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Definition

An n-category consists of objects, 1-morphisms between objects, 2-morphisms between 1-morphisms, ... , n-morphisms between $(n-1)$-morphisms. If k-morphisms exist for all natural numbers k, it is an ∞-category.

Strict and weak higher categories

If associativity and identities are defined strictly, then there is no problem with this definition.

However, many higher categories that arise in mathematics are not so strict.

Sometimes, for example, associativity only holds up to isomorphism, which then has to satisfy coherence conditions.

Keeping track of all these conditions becomes complicated!

There are many different definitions for what weak n-categories or weak ∞-categories should be, but we don't really know if they are equivalent to one another.

What has been more tractable has been the notion of $(\infty, 1)$-categories, or weak ∞-categories with all k-morphisms invertible for $k > 1$.

There are several ways to think about them, and their respective homotopy theories have been shown to be equivalent to one another.

More generally, one can define (∞, n)-categories, or weak ∞-categories with k-morphisms invertible for $k > n$.

Example: Cobordism categories

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But, we could also consider diffeomorphisms of these n-manifolds to get invertible $(n+1)$-morphisms, isotopies of the diffeomorphisms as $(n+2)$-morphisms, and so forth, to get an (∞, n)-category.

These examples appear in Lurie's proof of the Baez-Dolan Cobordism Hypothesis.

Basic case: $(\infty, 0)$-categories

It is often taken as a definition that $(\infty, 0)$-categories, which are just weak ∞-groupoids, are just topological spaces.

Why is this a sensible approach?

Given a topological space X, think of the points of X as objects and paths between points as morphisms. Paths are not strictly invertible, but they are weakly invertible.

Then homotopies between paths form 2-morphisms, and again they are weakly invertible.

We can take homotopies between homotopies as 3-morphisms, and so forth.

Since $(\infty, 0)$-categories are just topological spaces, they have a nice homotopy theory.

The weak equivalences are weak homotopy equivalences (maps which induce isomorphisms on all homotopy groups).

Often it is more useful to use the equivalent homotopy theory of simplicial sets instead.

Higher categories as enriched categories

A general principle in higher category theory is that a strict n-category is a category enriched in $(n-1)$-categories.

In other words, an n-category has objects and an $(n-1)$-category of morphisms between any two objects.

It is expected that this principle should hold for weak versions, too.

So, our first approach to $(\infty, 1)$-categories is that they can be defined to be categories enriched in $(\infty, 0)$-categories.

Often these are called *topological categories* or *simplicial categories*.

Topological categories as $(\infty, 1)$-categories

How can we think of a topological category as an $(\infty, 1)$-category?

Now, we have objects and topological spaces of morphisms between them, $\mathrm{Map}(x, y)$.

The points in these spaces now define 1-morphisms, and there is no reason to suppose they are invertible.

The paths, which are invertible, are now 2-morphisms.

Higher morphisms are defined as before, but everything is shifted up by one due to the enrichment.

We still have a nice way to do homotopy theory with simplicial (or topological) categories.

Definition

A simplicial functor $f : \mathcal{C} \to \mathcal{D}$ is a *Dwyer-Kan equivalence* if:

- $\mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{D}}(fx, fy)$ is a weak equivalence of simplicial sets, and
- $\pi_0\mathcal{C} \to \pi_0\mathcal{D}$ is an equivalence of categories.

Theorem (B)

There is a model structure on the category of simplicial categories with weak equivalences the Dwyer-Kan equivalences.

What about $(\infty, 2)$-categories?

Can we continue this process of enrichment?

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However, there are two main problems with this approach.

- The structure becomes too rigid for many examples. Since each enrichment has strict composition, there is still a lot of structure.
- There is no longer a nice homotopy theory. While the model structure on simplicial sets is very nice, the model structure on simplicial categories doesn't have as many nice properties: not cartesian.

Rethinking $(\infty, 1)$-categories

So, we might go back to $(\infty, 1)$-categories and ask if there are other ways to think about them which are still equivalent.

There are in fact many other models for them

- quasi-categories
- Segal categories
- complete Segal spaces
- categories with weak equivalences

Today, we'll focus on complete Segal spaces.

Segal spaces

We're going to consider simplicial spaces, or simplicial objects in the category of simplicial sets (also called bisimplicial sets).

Definition

A (Reedy fibrant) simplicial space $W \colon \mathbf{\Delta}^{op} \to \mathcal{SS}ets$ is a *Segal space* if the Segal maps

$$\varphi_k \colon W_k \to \underbrace{W_1 \times_{W_0} \cdots \times_{W_0} W_1}_{k}$$

are weak equivalences of simplicial sets.

This Segal condition can be thought of as giving a weak composition.

Segal categories

Definition

A *Segal category* is a Segal space such that W_0 is discrete.

Theorem (Pelissier, B)

There is a model structure on the category of simplicial spaces with W_0 discrete such that the fibrant objects are Segal categories.

Complete Segal spaces

In a Segal space W, we can think of its "objects" as being the set $W_{0,0}$.

It then has "mapping spaces" between objects defined as the homotopy fiber

$$\begin{tikzcd} \operatorname{map}(x,y) \arrow[r] \arrow[d] & W_1 \arrow[d] \\ \{(x,y)\} \arrow[r] & W_0 \times W_0. \end{tikzcd}$$

The image of the degeneracy map $W_0 \to W_1$ can be thought of as the "identity maps", and composition can be defined up to homotopy, so we can also define "homotopy equivalences" which form a subspace of W_1 which we denote W_h.

Definition

A Segal space W is *complete* if $W_0 \to W_h$ is a weak equivalence.

Another way to think about this is that W_0 is a moduli space for equivalences in W_1.

Complete Segal spaces as $(\infty, 1)$-categories

Theorem (Rezk)

There is a model structure on the category of simplicial spaces in which the fibrant objects are precisely the complete Segal spaces. Furthermore, this model structure is cartesian.

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Theorem (B)

The model structures on simplicial categories, Segal categories, and complete Segal spaces are equivalent to one another.

The Θ-construction

To understand how to generalize complete Segal spaces to models for higher (∞, n)-categories, we first need to generalize $\mathbf{\Delta}$.

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Let $\mathcal{C}$ be a category.

Define a new category $\Theta\mathcal{C}$ to have objects

$$[m](c_1, \ldots, c_m)$$

where $[m]$ is an object of $\mathbf{\Delta}$ and $c_1, \ldots, c_m$ objects of $\mathcal{C}$.

A morphism

$$[m](c_1, \ldots, c_m) \to [p](d_1, \ldots, d_p)$$

is given by $\delta \colon [m] \to [p]$ and $f_{ij} \colon c_i \to d_j$ for certain choices of i and j.

Example of a morphism in $\Theta\mathcal{C}$

An example of a morphism

$$[4](c_1, c_2, c_3, c_4) \rightarrow [3](d_1, d_2, d_3)$$

is given by

$$\begin{tikzcd}
0 \arrow[r, "c_1"] \arrow[d] & 1 \arrow[r, "c_2"] \arrow[dr] & 2 \arrow[r, "c_3"] \arrow[d] & 3 \arrow[r, "c_4"] \arrow[d] & 4 \arrow[dl] \\
0 \arrow[r, "d_1"] & 1 \arrow[r, "d_2"] & 2 \arrow[r, "d_3"] & 3 &
\end{tikzcd}$$

with morphisms $c_1 \rightarrow d_1$, $c_1 \rightarrow d_2$, and $c_3 \rightarrow d_3$ in $\mathcal{C}$.

The categories Θ_n

Define $\Theta_0 = *$, the category with one object and an identity morphism.

Then inductively define

$$\Theta_n = \Theta\Theta_{n-1}.$$

Notice that $\Theta_1 = \mathbf{\Delta}$.

To get an idea of the objects in Θ_2, consider the example of $[4]([2],[3],[0],[1])$.

We can think of labeling the arrows of [4] as follows:

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\end{tikzcd}
$$

These labels can also be interpreted as strings of arrows, so we can represent this object as

$$
\begin{tikzcd}[column sep=large]
0 \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2.5ex, "\Downarrow"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow"] & 1 \arrow[r, bend left=80] \arrow[r, bend left=25] \arrow[r, bend right=25] \arrow[r, bend right=80] \arrow[r, phantom, shift left=4.5ex, "\Downarrow"] \arrow[r, phantom, "\Downarrow"] \arrow[r, phantom, shift right=4.5ex, "\Downarrow"] & 2 \arrow[r] & 3 \arrow[r, bend left=40] \arrow[r, bend right=40] \arrow[r, phantom, "\Downarrow"] & 4.
\end{tikzcd}
$$

Θ_n-spaces

Just as we considered simplicial spaces, or functors $\mathbf{\Delta}^{op} \to \mathcal{SS}ets$, we want to consider functors

$$X \colon \Theta_n^{op} \to \mathcal{SS}ets.$$

We can again impose Segal and completeness conditions on such functors, in which case we call them Θ_n-*spaces*.

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Theorem (Rezk)

There is a cartesian model structure on the category of functors $X \colon \Theta_n^{op} \to \mathcal{SS}ets$ such that the fibrant objects are precisely the Θ_n-spaces.

Now, using a theorem of Lurie, have the following result.

Theorem (B-Rezk)

There is a model structure on the category of small categories enriched in Θ_n-spaces.

Now, if we enrich in Θ_{n-1}-spaces, we want this model structure to be equivalent to the one for Θ_n-spaces.

To prove this result, we need a string of equivalences between several model categories.

Theorem (B-Rezk)

There are Quillen equivalences

$$(\Theta_{n-1}Sp) - Cat \leftrightarrows (\Theta_{n-1}Sp)^{\mathbf{\Delta}^{op}}_{disc,Se,f} \rightleftarrows (\Theta_{n-1}Sp)^{\mathbf{\Delta}^{op}}_{disc,Se,c}.$$

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This chain should be extended to

$$(\Theta_{n-1}Sp)^{\mathbf{\Delta}^{op}}_{disc,Se,c} \rightleftarrows (\Theta_{n-1}Sp)^{\mathbf{\Delta}^{op}}_{Se,Cpt,c} \rightleftarrows \Theta_n Sp.$$