A homotopy-theoretic view of Bott–Taubes integrals

Robin Koytcheff

Stanford University

November 8, 2009

Intro

- We will construct cohomology classes in spaces of knots $\mathcal{K}$ where $\mathcal{K} = \mathrm{Emb}(S^1, \mathbb{R}^n)$ or $\mathrm{Emb}(\mathbb{R}, \mathbb{R}^n)$ ($n \geq 3$).
- Connected components of $\mathcal{K}$ correspond to isotopy classes of knots, so for $n = 3$, $H^0\mathcal{K} = \{\text{knot invariants}\}$.
- We'll make use of a bundle

$$\begin{tikzcd} F \arrow[r] & E \arrow[d] \\ & \mathcal{K} \end{tikzcd}$$

whose fiber F is a compactified configuration space.

Intro

- Bott & Taubes found knot invariants (elements in $H^0(\mathrm{Emb}(S^1, \mathbb{R}^3))$ by integrating differential forms along the fiber of this bundle over $\mathrm{Emb}(S^1, \mathbb{R}^3)$.
- Their techniques give one way to construct Vassiliev (i.e., finite-type) invariants (D. Thurston; Altschuler & Freidel; Volić), and they have been used to construct classes in $H^*(\mathrm{Emb}(S^1, \mathbb{R}^n); \mathbb{R})$ for $n \geq 4$.
- We will "integrate along fiber" via algebraic topology to get elements of $H^*(\mathcal{K})$ with *arbitrary* coefficients.
- We also prove a product formula with respect to connect-sum for these classes.

Configuration space and its compactification

- Configuration space of q points in a space X is $C_q(X) := \{(x_1, ..., x_q) \in X^q | x_i \neq x_j \forall i \neq j\}$, an open subset of X^q.

  Notice $X \hookrightarrow Y$ induces a map $C_q(X) \hookrightarrow C_q(Y)$.

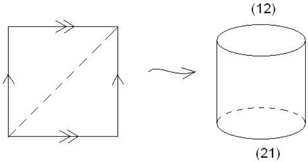
- For integration, we need a compact manifold. Let $C_q[M]$ be the Fulton–MacPherson/Axelrod–Singer compactification of $C_q(M)$ (where $M =$ compact manifold or $\mathbb{R}^N$).

  Intuitively, this compactification keeps track of the directions of collisions of points and their relative rates of approach.

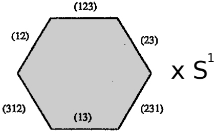
  Compactified configuration spaces are smooth manifolds with corners, and homotopy type is unchanged by compactification.

Examples of compactified configuration spaces

- $C_2[S^1] = S^1 \times I$:



- $C_3[S^1]$ is the product of S^1 with a hexagon, as shown below:



- $C_1[\mathbb{R}^n] = D^n$ closed disk.

Bundle over knot space

Let $\mathcal{K} = \mathrm{Emb}(S^1, \mathbb{R}^3)$.

Definition

$E = E_{q,t} \to \mathcal{K}$ is a bundle whose total space is the pullback below:

$$\begin{tikzcd} E_{q,t} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C_{q+t}[\mathbb{R}^3] \arrow[d] \\ C_q[S^1] \times \mathcal{K} \arrow[r] & C_q[\mathbb{R}^3] \end{tikzcd}$$

Bundle $E_{q,t} \to \mathcal{K}$ given by following left-hand map by projection to $\mathcal{K}$.

Fiber $F = F_{q,t}$ over a knot K is the pullback

$$\begin{tikzcd} F_{q,t} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C_{q+t}[\mathbb{R}^3] \arrow[d] \\ C_q[S^1] \arrow[r, "\text{embed by } K"] & C_q[\mathbb{R}^3] \end{tikzcd}$$

and is a (finite-dim'l) smooth manifold with corners (Bott & Taubes).

Integration along the fiber

- Bott and Taubes integrated differential forms coming from $H^*(C_{q+t}[\mathbb{R}^3])$ along fibers of $E_{q,t} \to \mathcal{K}$ for various q, t:

$$\int_F : \Omega^p_{dR}(E) \to \Omega^{p-\dim F}_{dR}(\mathcal{K})$$

  They constructed an element in $\Omega^0_{dR}(\mathcal{K})$.

- Not immediate that the resulting form is closed since F has boundary, i.e., Stokes' Theorem gives

$$d \int_F = \int_F d + \int_{\partial F}.$$

  Bott & Taubes show that their 0-form is closed, hence is in $H^0\mathcal{K}$ and is a knot invariant.

- Others built upon this work to give a construction of all Vassiliev invariants and to give classes in $H^*(\mathrm{Emb}(S^1, \mathbb{R}^n); \mathbb{R})$ ($n > 3$) in arbitrarily high degrees $*$.

Pontrjagin–Thom construction for fiber bundles

- Suppose $F \longrightarrow E \xrightarrow{p} B$ is a fiber bundle of smooth compact manifolds. Embed $\overline{e} : E \hookrightarrow \mathbb{R}^N$ for some N, and consider the resulting embedding e:

$$e : E \xhookrightarrow{(p,\overline{e})} B \times \mathbb{R}^N$$

- Quotienting by the complement of a tubular neighborhood η of E gives the Pontrjagin–Thom collapse map

$$B \times \mathbb{R}^N \longrightarrow \Sigma^N B_+ \xrightarrow{\tau} E^\nu$$

where $\nu =$ normal bundle of e, $E^\nu = D(\nu)/S(\nu) =$ the Thom space of ν.

- Together with the Thom $\cong$ and suspension $\cong$, τ induces a map

$$H^* E \to H^{* - \dim F} B$$

which agrees with integration along the fiber in de Rham cohomology.

Pontrjagin–Thom for bundle over knot space

- Want to do $\int_F$ via Pontrjagin–Thom for the bundle $E \to \mathcal{K}$, where $\mathcal{K} = \mathrm{Emb}(S^1, \mathbb{R}^n)$ or $\mathrm{Emb}(\mathbb{R}, \mathbb{R}^n)$.
- Issue: $E = E_{q,t}$ has boundary (even corners) coming from the fiber $F = F_{q,t} \subset C_{q+t}[\mathbb{R}^3]$.
- We can construct an embedding e which preserves the corner structure:

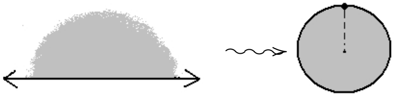
$$\begin{tikzcd} E \arrow[r, hook] \arrow[rr, bend left, "e"] & \mathcal{K} \times C_{q+t}[\mathbb{R}^3] \arrow[r, hook] & \mathcal{K} \times [0,\infty)^L \times \mathbb{R}^N \end{tikzcd}$$

  Thus we can make sense of the normal bundle of e.
- We get a Thom collapse map

$$\mathcal{K} \times [0,\infty)^L \times \mathbb{R}^N \xrightarrow{\tau} E^{\nu} \;.$$

Homotopy-theoretic Bott–Taubes integrals

One-point compactification of $[0,\infty)^L \times \mathbb{R}^N$ gives $C^L S^N$. E.g., for L=N=1,

$[0,\infty) \times \mathbb{R} =$  $= D^2 = C(S^1)$

Quotient by ∂ to get S^{L+N} (then susp. $\cong$ gives correct degree shift). Let $N \to \infty$:

Theorem (K)

There is a map of spectra

$$\Sigma^\infty \mathcal{K}_+ \xrightarrow{\tau} E^\nu / \partial E^\nu \ .$$

which in cohomology induces a map analogous to integration along the fiber:

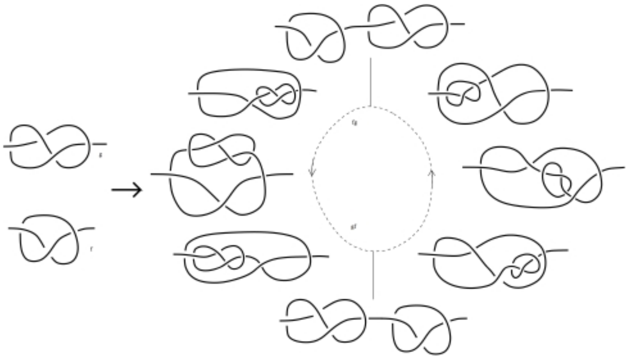
$$H^{*+\dim F}(E, \partial E) \to H^*(\mathcal{K})$$

Works with arbitrary coefficients!

Connect-sum of knots is commutative on $\pi_0(\mathcal{K})$

Now let $\mathcal{K} = \mathrm{Emb}(\mathbb{R}, \mathbb{R}^3)$.

Budney defined an action of the little 2-cubes operad $\mathcal{C}_2$ on $\mathcal{K}$, which is motivated by the commutativity of connect-sum on isotopy classes of knots; i.e., the operation on $\mathcal{K}$ is homotopy-commutative.



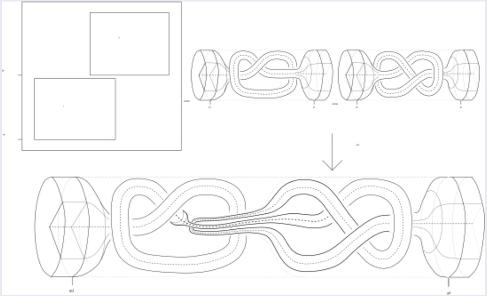
Little 2-cubes action on space of long knots in $\mathbb{R}^3$

Budney defined the action on the space of "fat long knots" = $\{\text{embeddings } f : \mathbb{R} \times D^2 \hookrightarrow \mathbb{R} \times D^2 | f = id \text{ outside } [-1,1] \times D^2\}$.

The subspace of all f such that $\ell k(f|_{\mathbb{R}\times(0,0)}, f|_{\mathbb{R}\times(0,1)}) = 0$, which is homotopy equivalent to the space of long knots.

Definition (Budney)

The $\mathcal{C}_2$-action on $\mathcal{K}$ is defined as follows:



Homology of space of long knots in $\mathbb{R}^3$

- Budney further showed that $\mathcal{K}$ is the free 2-cubes object on the space of prime knots $\mathcal{P}$:

$$\mathcal{K} \simeq \mathcal{C}_2(\mathcal{P} \sqcup \{*\}) := \coprod_{j=0}^{\infty} \mathcal{C}_2(j) \times_{\Sigma_j} \mathcal{P}^j$$

- Combining this with results of F. Cohen's gave a description of $H_*\mathcal{K}$:

  Essentially, $H_*\mathcal{K}$ is generated by taking products of, and applying certain homology operations to, classes in $H_*\mathcal{P}$.
- E.g., $H_*(\mathcal{K};\mathbb{Q})$ is generated as an algebra by $\{\ ,\ \}$'s of elements in $H_*(\mathcal{P};\mathbb{Q})$, where $\{\ ,\ \}$ is the Browder operation given by

$$\begin{aligned} H_1(\mathcal{C}_2(2)) \otimes H_m\mathcal{K} \otimes H_n\mathcal{K} &\to H_{m+n+1}\mathcal{K} \\ [S^1] \otimes \alpha \otimes \beta &\mapsto \{\alpha, \beta\} \end{aligned}$$

- For $\mathbb{Z}/p$ coefficients, need Dyer–Lashof operation(s) Q_i (and mod-p Bockstein for odd p).

Towards a product formula

Let $\beta \in H^*E$, let $a_1, a_2 \in H_*\mathcal{K}$, and let $\xi_* : H_*\mathcal{K} \otimes H_*\mathcal{K} \to H_*\mathcal{K}$ be any homology operation.

- By duality of a product in homology with the appropriate coproduct in cohomology,
$$\langle \tau^* \beta, \xi_*(a_1 \otimes a_2) \rangle = \langle \xi^* \tau^* \beta, a_1 \otimes a_2 \rangle$$
- Together with the Budney–Cohen result, this motivates the goal of calculating the evaluation of a “Bott–Taubes class” $\tau^*\beta$ on any element in $H_*\mathcal{K}$ in terms of its evaluations on elements in $H_*\mathcal{P}$.

We lift the space-level connect-sum $\mu : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ to a multiplication

$$E_{q,t}/\partial E_{q,t} \times E_{r,s}/\partial E_{r,s} \to E_{q+r,t+s}/\partial E_{q+r,t+s}.$$

Product formula

Showing μ commutes with Thom collapse maps τ gives a map of *ring* spectra

$$\bigvee_{q,t\in\mathbb{N}} \Sigma^\infty \mathcal{K}_+ \to \bigvee_{q,t\in\mathbb{N}} E_{q,t}/\partial E_{q,t}$$

and also

$$\langle \tau^*\beta, \mu_*(a_1 \otimes a_2)\rangle = \langle \mu^*\tau^*\beta, a_1 \otimes a_2\rangle = \langle \tau^*\mu^*\beta, a_1 \otimes a_2\rangle$$

Theorem (K)

We can express an evaluation on a "Bott–Taubes class" in terms of evaluations on "smaller classes":

$$\langle \tau^*\beta, \mu_*(a_1 \otimes a_2)\rangle = \sum_i \langle \eta_i, a_1\rangle\langle \zeta_i, a_2\rangle$$

We can compute the η_i and ζ_i explicitly in terms of $H^*(C_{q+t}[\mathbb{R}^3])$.

Obstruction to lifting 2-cubes action

Lifting the 2-cubes action on $\mathcal{K}$ to the $E_{q,t}/\partial E_{q,t}$ would greatly help attain our goal of computing the evaluation of a Bott–Taubes class on any homology class in terms of evaluations on prime knot homology classes.

E.g., replace μ_* and μ^* in

$$\langle \tau^* \beta, \mu_*(a_1 \otimes a_2) \rangle = \langle \tau^* \mu^* \beta, a_1 \otimes a_2 \rangle$$

by bracket $\{,\}$ and an appropriate dual map in cohomology. However...

Proposition (K)

The given lifting of $\mu : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ to the $E_{q,t}/\partial E_{q,t}$ does not extend to a little 2-cubes action.

Sketch proof of Proposition: Since we consider *ordered* configurations, the multiplication on the $E_{q,t}/\partial E_{q,t}$ is noncommutative. □

Last slide

For more info, see

R. Koytcheff, "A homotopy-theoretic view of Bott–Taubes integrals and knot spaces". *Algebr. and Geom. Top.* 9: 3, pp. 1467-1501, 2009. Also [arXiv:0810.1785].

Email: robink@math.stanford.edu
Web: http://math.stanford.edu/∼robink

Thanks for your attention!