# Unstable Vassiliev theory

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# The Vassiliev spectral sequence (far too quickly)

Let $\mathcal{K}$ be the space of long knots in $\mathbb{R}^3$.

Goal: Understand $H^0(\mathcal{K})$.

Plan: (Vassiliev [3]) Study instead the space of singular maps.

1. Model $\mathcal{K}$ by finite-dimensional knot spaces $\mathcal{K}_m$
2. Blow up the complementary *discriminants* $\Sigma_m$.
3. Filter $\tilde{\Sigma}_m$ by *complexity*.
4. Analyze the combinatorics of the spectral sequence of this filtration is a stable range.
5. Apply Alexander duality to get knot invariants.

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# Outline

# Plumbers' curves

Consider the spaces $P_m$ of *plumbers' curves of m-moves* [2]. These are maps $\phi : [0,1] \to [0,1]^3$ which satisfy

- $\phi(0) = (0,0,0)$, $\phi(1) = (1,1,1)$,
- $\phi$ travels parallel to coordinate axes, alternating in the order $(x, y, z)$, and
- $\phi$ has $3m$ segments (or, *pipes*) in *m moves*.

Two pipes are *distant* if separated by more than two pipes, and a plumbers' curve is *singular* if distant pipes intersect.

The collection of non-singular plumbers' curves is the space $K_m$ of *plumbers' knots*, and its complement $S_m$ is the *discriminant*.

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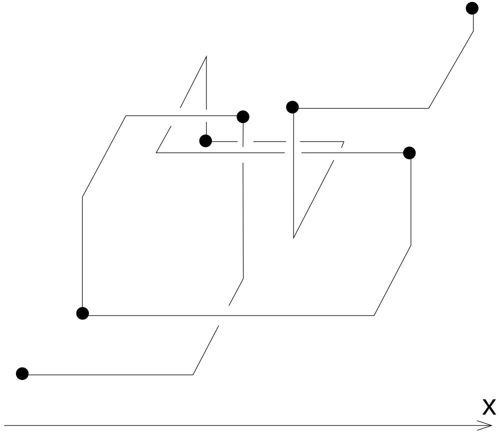
# A plumbers' knot of 6 moves



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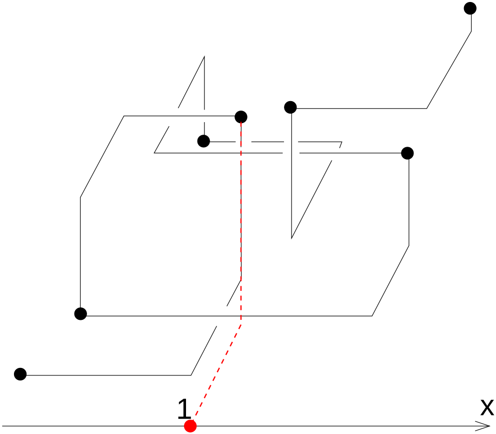

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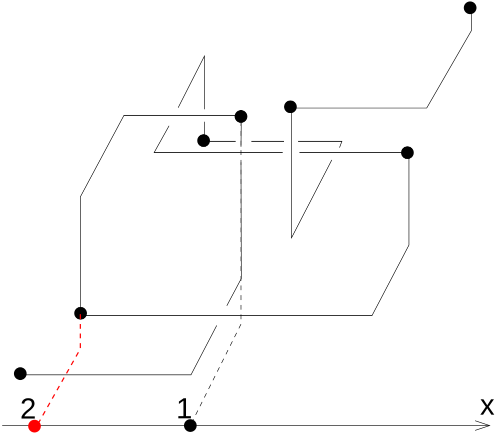

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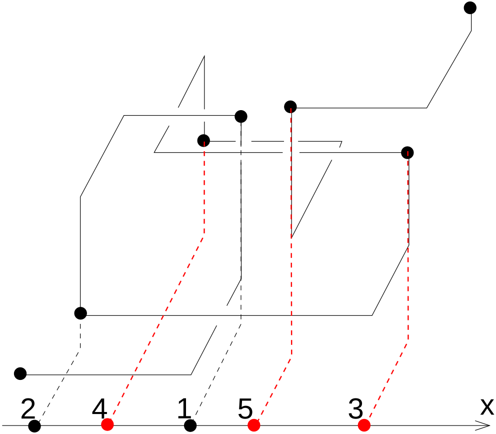

# Features of plumbers' knots

1. Combinatorial cell structure $\textsc{Cell}_\bullet(P_m)$ for each $m$.
2. $\textsc{Cell}_\bullet(S_m) \subseteq \textsc{Cell}_\bullet(P_m)$ as a closed subcomplex.

   Get an algorithm which classifies components of $K_m$. For example, $K_5$ has 7 components: the unknot and three of each trefoil, $K_6$ has 49 components and $K_7$ has 1008.
3. The spaces $P_m$ fit into a directed system of inclusions, inducing such on $K_m$ and $S_m$.

**Theorem**

$\pi_0(\mathrm{colim} K_m) \cong \pi_0(\mathcal{K})$

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# The combinatorial structure of $S_m$

## Definition

Let $\mathcal{S}_m$ be the category whose objects are non-empty elements of $\mathcal{P}\left(\binom{[\mathbf{m-1}]}{2} \times \{x, y, z\}\right)$ with morphisms given by inclusions.

Objects in this category correspond to collections of coordinate equalities.

## Definition

Let $B_m : \mathcal{S}_m \to \mathbf{Top}$ be the contravariant functor given by $B_m(\mathbf{C}) = \{\phi \in S_m : \phi \text{ respects } \mathbf{C}\}$.

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$\mathrm{colim} B_m \cong S_m$

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# Blowing up $S_m$

In order for Alexander duality to "see" singularity data, cells must be in the proper codimension.

**Definition (Blowup of the discriminant)**

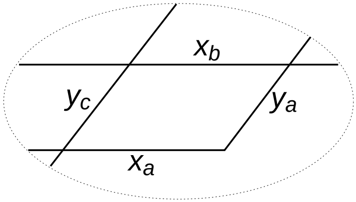
$\tilde{S}_m = \mathrm{hocolim} B_m$

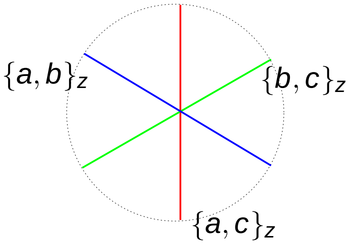
**Lemma**

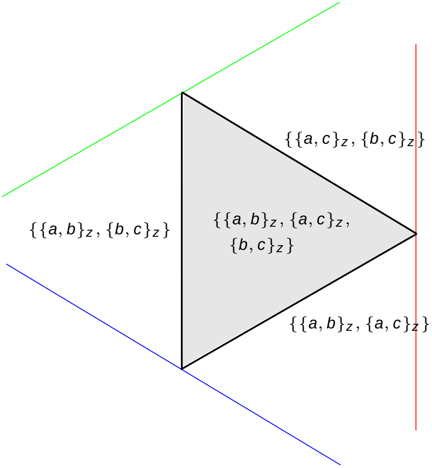
$\tilde{S}_m \simeq S_m$

Moreover, we can lift the cell structure on $S_m$ to one on $\tilde{S}_m$, retaining (and enriching) the combinatorics.

# A cell in $\tilde{S}_m$



# Derivatives of an invariant of plumbers' knots

Let $[\alpha] \in \bar{H}^0(K_m)$ and $\tilde{\mathbf{e}} \in \mathbf{C}_{3m-4}(\tilde{S}_m)$.

## Definition (Vassiliev derivative)

$$d_{\tilde{e}}([\alpha]) = \begin{cases} [\alpha](b) - [\alpha](a) & \tilde{\mathbf{e}} \text{ separates some pair } a, b \in H_0(K_m) \\ 0 & \text{else} \end{cases}$$

## Theorem

The lift to $\tilde{S}_m$ of the Alexander dual to $[\alpha]$ has a chain representative given by $\tilde{\alpha}^\vee = \sum_{\tilde{e} \in C_{3m-4}(S_m)} (-1)^{\sigma(\tilde{e})} d_{\tilde{e}}([\alpha])\tilde{e}$.

Of course, this representative is only well defined up to a choice of boundary.

# Taylor's Theorem

Note that this theorem gives information for any singular map, in contrast to Vassiliev's acyclicity results.

**"Taylor's Theorem"**

There exists a canonical Vassiliev derivative for plumbers' knot invariants associated to each singularity type for plumbers' knots.

**Corollary**

Each $[\alpha] \in \bar{H}^0(K_m)$ is completely determined by its collection of Vassiliev derivatives.

# The filtration on $\tilde{S}_m$

We require a filtration on $\tilde{S}_m$ which agrees with the classical Vassiliev filtration on the singularities he considers.

First guess: filter by the number of distant pipes which intersect.

Correction: We must not increase the filtration for "going around corners" or "$n$-fold points becoming $(n+1)$-fold points".

**(Most of a) Definition**

The *complexity*, $c(\phi)$, of a plumbers' knot $\phi$ is given by (something ugly and combinatorial). Let $F_p(\tilde{S}_m) = \{\phi \in S_m : c(\phi) \geq p\}$.

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# Collapse!

By reindexing, we can consider the homology spectral sequence of this filtration as a cohomology spectral sequence, $E_r^{*,*}(m)$, converging to $H^*(K_m)$.

**Theorem**

$E_r^{*,*}(m)$ collapses at the $E_2$ page.

We believe this can be improved to show collapse at the $E_1$ page.

**Remarks**

This gives us an honest inverse system of spectral sequences, each of which converges to the complete cohomology of the space in question.

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# Current directions

1. Any knot invariant has a restriction to each $K_m$. What are the Vassiliev derivatives of these restrictions and how do they evolve in the inverse system? (Are integer coefficient weight systems "integrable"?)
2. Which choices of derivatives produce invariants of plumbers' knots? (What are "unstable weight systems" for plumbers' knot invariants?)
3. There is a splitting of plumbers' knot invariants (over $\mathbb{Q}$) into "stable" and "unstable" summands. Do unstable invariants contribute to the inverse limit? (Do finite-type invariants distinguish all knots?)
4. Baldridge and Lowrance's cube diagrams [1] compute knot Floer homology and can be considered a class of plumbers' knots. What can this connection tell us?

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