Spaces with algebraic structure

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What is a space with algebraic structure?

A topological space has algebraic structure if, in addition to being a topological space, it is also an algebraic object, such as a monoid, a group, or a ring, for example, and the two structures are compatible.

Definition

A *topological group* G is a topological space which is also a group, such that the multiplication map

$$G \times G \to G$$

and the inverse map

$$G \to G$$

are continuous maps.

Examples

There are lots of examples of topological groups. A few of them are $\mathbb{R}$, $\mathbb{C}$, and $S^1 \subseteq \mathbb{C}$.

Lie groups, or topological groups which also happen to be smooth manifolds, are studied from many perspectives.

An interesting fact about topological groups is that there are many other structures which end up being equivalent to them. Today, we are going to look at some that are equivalent from the viewpoint of homotopy theory.

Free groups

First, let's look at some particular kinds of groups, namely, free groups, and some properties of them.

To get a free group, we assume that we have an identity, plus at least one other element, and we build up a group in the simplest possible way: by giving each element (called a generator) an inverse, and then combining all the elements in every possible way. We have the "relations" imposed on us by the definition of a group (so the identities and inverses behave as they should) but no others.

The free group on one generator, say x, is isomorphic to $\mathbb{Z}$, since it looks like

$$\cdots x^{-2}, x^{-1}, e, x, x^2, x^3, \cdots .$$

Free groups on more generators

Let's look at the free group on 2 generators, where we call the generators x and y. Building elements of the group from x, we get elements as before:

$$\cdots x^{-2}, x^{-1}, e, x, x^2, x^3, \cdots$$

and similarly for y:

$$\cdots y^{-2}, y^{-1}, e, y, y^2, y^3, \cdots .$$

But, we can also combine to get elements like

$$xy, \quad yx, \quad x^{-1}y, \quad x^3y^{-5}x^{-1}, \quad xy^8x^{-2}y^{-1}x^{19},$$

and so forth. We say that the elements of this group are *words* in x and y, and their inverses.

Notice that this group is no longer abelian. We write it as $\mathbb{Z} * \mathbb{Z}$.

A diagram of finitely generated free groups

We can take free groups on any number of generators.

In fact, here we want to look at all of them at once. We can put all the finitely generated free groups in a diagram, such that the maps between them are all possible group homomorphisms between them:

$$e \leftrightarrows F_1 \Leftrightarrow F_2 \Leftrightarrow F_3 \Leftrightarrow \cdots .$$

Notice that after F_1, there are actually lots and lots of maps between the groups, but it is hard to draw.

And on to topological groups

Now, we want to get a diagram that looks almost the same as the one above, but with topological spaces instead of free groups, and with all the arrows reversed:

$$* \rightleftarrows X_1 \Leftrightarrow X_2 \Leftrightarrow X_3 \Leftrightarrow \cdots .$$

We are also going to impose the condition that X_n is homeomorphic to $(X_1)^n$ for all $n \geq 2$.

These maps, just like the ones between free groups but reversed, and the product condition on the spaces are enough to give the space X_1 the structure of a topological group.

(For the experts: We are taking a product-preserving functor from the opposite of the category described on the previous slide to the category of topological spaces.)

Homotopy diagrams

But, I'm a homotopy theorist, so I don't like requiring that X_n be homeomorphic to $(X_1)^n$. I'd rather just ask that the two spaces be homotopy equivalent. Replacing homeomorphisms with homotopy equivalences in the above diagram of topological spaces gives X_1 the structure of a topological group up to homotopy.

While it might be expected that would give a weaker structure, it actually does not; these homotopy diagrams are also equivalent to topological groups.

Theorem (Badzioch)

Given a homotopy diagram as described above, one can actually "find" a strict diagram equivalent to it.

Why so many maps?

You might be curious, however, why we needed to have so many maps between our spaces. After all, don't we just need to define our group operation and make sure it is associative?

In fact, we don't actually need all the maps arising as homomorphisms of free groups to get a group structure on X_1.

For simplicity, I'm going to explain how to get a simpler diagram for monoids; it isn't too much harder to get the inverses in there later.

A simpler diagram for monoids...

Rather than building a diagram from free groups, we're going to build a diagram of finite ordered sets, and order-preserving maps between them:

Taking a diagram of spaces with these arrows but reversed, and such that $X_n \simeq (X_1)^n$, gives X_1 the structure of a topological monoid up to homotopy.

...and for groups

To get a similar diagram for groups, we need to have some way of encoding inverses. We can do so by putting a "flip" map at each level:

Theorem (B)

These diagrams are equivalent to the homotopy diagrams given by maps of free groups, and hence to topological groups.

Why not just use the original definition?

This diagram viewpoint is not particularly useful for investigating a particular topological group. What it is good for is the study of all topological groups and how they interact with one another.

When looking at groups, we usually specify a "basis", in that we define what we get when we apply the group operation to any two specific elements. For topological groups, we also need to understand what the topology is.

What the diagrams are good for

With the diagram approach, we don't have to understand what the precise group operation is, or even what the topology is. We know that we have some collection of topological spaces, and that the structure we have imposed on them will make one of those spaces have a group structure.

Thus, we can consider the collection of all topological groups in a systematic way and not worry about what a given one looks like, precisely.

Thus, the diagram approach is also useful when we want to consider interactions between topological groups and other kinds of mathematical structures.

A generalization and comparison

This approach to topological groups can be generalized to topological groupoids, which are topological spaces with an operation on them, but in which only certain elements can be composed with one another. The following theorem can be proved in a few different ways.

Theorem (Dwyer-Kan, B)

Topological groupoids are equivalent to topological spaces.

Conclusion

There are actually many, many more ways of looking at topological groups from the viewpoint of homotopy theory. If you are interested in learning more, see my slides for the talk "Thirteen ways of looking at a topological group," which can be found on my webpage:

http://www.math.ucr.edu/~jbergner/GroupTalk.pdf