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Rigidification of Algebras over Multi-Sorted Algebraic Theories

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http://www.nd.edu/˜jbergner/TheoryTalk.pdf

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Algebraic Theories

Definition 1. An *algebraic theory* is a small category with objects $T_0, T_1, T_2, \ldots$ such that each T_n can be written as the product $(T_1)^n$.

Proposition 2. *(Lawvere) Given an algebraic category $\mathcal{C}$ with free objects, there is an algebraic theory $\mathcal{T}$ such that the category of product-preserving functors $\mathcal{T} \rightarrow \mathcal{S}ets$ is equivalent to $\mathcal{C}$.*

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Example 3. Let $\mathcal{M}$ be the category of monoids. Consider the full subcategory of $\mathcal{M}$ whose objects are the finitely generated free monoids. Denote by $\mathcal{T}_M$ the opposite of this category. The category of product-preserving functors $\mathcal{T}_M \to \mathcal{S}ets$ is equivalent to $\mathcal{M}$. Hence, we call $\mathcal{T}_M$ the *theory of monoids.*

We can also consider product-preserving functors $\mathcal{T}_M \to \mathcal{S}paces$. The category of such functors is equivalent to the category of topological monoids.

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Definition 4. Let $\mathcal{T}$ be an algebraic theory. A *(strict) $\mathcal{T}$-algebra* is a product-preserving functor $A : \mathcal{T} \to \mathcal{S}paces$.

One might ask, however, about the case when we only have products preserved up to homotopy.

Definition 5. A *homotopy $\mathcal{T}$-algebra* is a functor $X : \mathcal{T} \to \mathcal{S}paces$ which preserves products up to homotopy. In other words, $X(T_n) \simeq X(T_1)^n$ is a weak equivalence.

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There is a model category structure $\mathcal{A}lg^{\mathcal{T}}$ on the category of all $\mathcal{T}$-algebras. There is also a homotopy $\mathcal{T}$-algebra model category structure $h\mathcal{A}lg^{\mathcal{T}}$.

We can use the following result to "rigidify" any homotopy $\mathcal{T}$-algebra:

Theorem 6. *(Badzioch) There is a Quillen equivalence of model categories between $\mathcal{A}lg^{\mathcal{T}}$ and $h\mathcal{A}lg^{\mathcal{T}}$.*

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An Application

Consider again the theory of monoids $\mathcal{T}_M$. We have seen that a topological monoid is equivalent to a strict $\mathcal{T}_M$-algebra. By Theorem 6, we can consider it as a homotopy $\mathcal{T}_M$-algebra.

Definition 7. A *reduced Segal category* is a simplicial space X such that X_0 is the space consisting of a single point and such that for any $n \geq 2$, the Segal maps

$$\varphi_k : X_k \to (X_1)^k$$

are weak equivalences of spaces.

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There is a reduced Segal category model category structure $\mathcal{S}e\mathcal{C}at_*$.

Theorem 8. *(B) There is a Quillen equivalence of model categories between $h\mathcal{A}lg^{\mathcal{T}_M}$ and $\mathcal{S}e\mathcal{C}at_*$.*

Therefore, we can consider topological monoids to be equivalent, in some sense, to reduced Segal categories.

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Recall that a monoid is a category with one object. Thus, a topological monoid is a topological category with one object.

Definition 9. A *topological category* is a category with a space of morphisms between any two objects.

We would like to generalize the above result to topological categories, but we cannot use algebraic theories because in a general category, not all morphisms compose.

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Multi-Sorted Theories

Definition 10. A *multi-sorted algebraic theory* sorted by a set S is a small category with objects $T_{\underline{\alpha}^n}$ where $\underline{\alpha}^n =< \alpha_1, \ldots, \alpha_n >$ for $\alpha_i \in S$ such that

$$T_{\underline{\alpha}^n} \cong \prod_{i=1}^{n} T_{\alpha_i}$$

for each $n \geq 0$ and $\underline{\alpha}^n \in S^n$.

We will consider the example of the theory of categories with object set $\mathcal{O}$.

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Example 11. Let $\mathcal{O}$ be the set with two elements. The theory $\mathcal{T}_{\mathcal{O}\mathcal{C}at}$ has as objects the free categories with two objects.

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We can define strict and homotopy $\mathcal{T}$-algebras for a multi-sorted theory $\mathcal{T}$ just as we did for ordinary algebraic theories. Again, we have respective model category structures $\mathcal{A}lg^{\mathcal{T}}$ and $h\mathcal{A}lg^{\mathcal{T}}$.

Theorem 12. *(B) Let $\mathcal{T}$ be a multi-sorted algebraic theory. There is a Quillen equivalence of model categories between $\mathcal{A}lg^{\mathcal{T}}$ and $h\mathcal{A}lg^{\mathcal{T}}$.*

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Definition 13. Let $\mathcal{O}$ be a set. A *Segal $\mathcal{O}$-category* is a simplicial space X such that $X_0 = \mathcal{O}$ and such that for each $k \geq 2$ the Segal map $\varphi_k : X_k \to \underbrace{X_1 \times_{\mathcal{O}} \cdots \times_{\mathcal{O}} X_1}_{k}$ is a weak equivalence.

Again, for a fixed set $\mathcal{O}$, there is a model category structure $\mathcal{S}e\mathcal{C}at_{\mathcal{O}}$.

Theorem 14. *(B) Let $\mathcal{O}$ be a set. There is a Quillen equivalence between $h\mathcal{A}lg^{\mathcal{T}_{\mathcal{O}\mathcal{C}at}}$ and $\mathcal{S}e\mathcal{C}at_{\mathcal{O}}$.*

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Therefore, we can think of topological categories as being equivalent, in some sense, to Segal categories.

However, keeping the object set fixed is restrictive. Hence, we consider a model category structure $\mathcal{TC}$ on the category of all small topological categories and a Segal category model category structure $\mathcal{S}e\mathcal{C}at$.

Theorem 15. *(B) There is a Quillen equivalence of model categories between $\mathcal{TC}$ and $\mathcal{S}e\mathcal{C}at$.*

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