Homotopy-theoretic approaches to higher categories

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April 30, 2011

Idea of categories

Homotopy theories

Higher categories

In a category, we have morphisms between objects.
But what if you have functions between functions?
This gives the idea of a “2-morphism”.
Then, more generally could have $(n+1)$-morphisms between n-morphisms.

Definition

An n-category consists of objects, 1-morphisms between objects, 2-morphisms between 1-morphisms, ... , n-morphisms between $(n-1)$-morphisms. If k-morphisms exist for all natural numbers k, it is an ∞-category.

Strict and weak higher categories

If associativity and identities are defined strictly, then there is no problem with this "definition."

However, many higher categories that arise in mathematics are not so strict.

Sometimes, for example, associativity only holds up to isomorphism, which then has to satisfy coherence conditions.

Keeping track of all these conditions becomes complicated!

There are many different definitions for what weak n-categories or weak ∞-categories should be, but we don't really know if they are equivalent to one another.

Homotopical higher categories

What has been more tractable has been the notion of $(\infty, 1)$-categories, or weak ∞-categories with all k-morphisms invertible for $k > 1$.

There are several ways to think about them, and their respective homotopy theories have been shown to be equivalent to one another.

More generally, one can define (∞, n)-categories, or weak ∞-categories with k-morphisms invertible for $k > n$.

Example: Cobordism categories

Basic case: $(\infty, 0)$-categories

It is often taken as a definition that $(\infty, 0$-categories, which are just weak ∞-groupoids, are just topological spaces.

Why is this a sensible approach?

Given a topological space X, think of the points of X as objects and paths between points as morphisms.

Paths are not strictly invertible, but they are weakly invertible.

Then homotopies between paths form 2-morphisms, and again they are weakly invertible.

We can take homotopies between homotopies as 3-morphisms, and so forth.

Homotopy theory for $(\infty, 0)$-categories

Since $(\infty, 0)$-categories are just topological spaces, they have a nice homotopy theory.

The weak equivalences are weak homotopy equivalences (maps which induce isomorphisms on all homotopy groups).

Sometimes, it is more useful to use the equivalent homotopy theory of simplicial sets instead.

Simplicial sets

Higher categories as enriched categories

A general principle in higher category theory is that a strict n-category is a category enriched in $(n-1)$-categories.
In other words, an n-category has objects and an $(n-1)$-category of morphisms between any two objects.
It is expected that this principle should hold for weak versions, too.
So, our first approach to $(\infty, 1)$-categories is that they can be defined to be categories enriched in $(\infty, 0)$-categories.
Often these are called topological categories or simplicial categories.

Topological categories as $(\infty, 1)$-categories

How can we think of a topological category as an $(\infty, 1)$-category?
Now, we have objects and topological spaces of morphisms between them, $\mathrm{Map}(x, y)$.
The points in these spaces now define 1-morphisms, and there is no reason to suppose they are invertible.
The paths are now 2-morphisms, and now they are invertible.
Higher morphisms are defined as before, but everything is shifted up by one due to the enrichment.

The homotopy theory of simplicial categories

We still have a nice way to do homotopy theory with simplicial (or topological) categories.

Definition

A simplicial functor $f : \mathcal{C} \to \mathcal{D}$ is a *Dwyer-Kan equivalence* if:

- $\mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{D}}(fx, fy)$ is a weak equivalence of simplicial sets, and
- $\pi_0\mathcal{C} \to \pi_0\mathcal{D}$ is an equivalence of categories.

Theorem

There is a model structure on the category of simplicial categories with weak equivalences the Dwyer-Kan equivalences.

What about $(\infty, 2)$-categories?

Can we continue this process of enrichment?
We could define $(\infty, 2)$-categories as categories enriched in $(\infty, 1)$-categories.
However, there are two main problems with this approach.

- The structure becomes too rigid for many examples. Since each enrichment has strict composition, there is still a lot of structure.
- There is no longer a nice homotopy theory. While the model structure on simplicial sets is very nice, the model structure on simplicial categories doesn't have as many nice properties: not compatible with internal hom objects.

Rethinking $(\infty, 1)$-categories

So, we might go back to $(\infty, 1)$-categories and ask if there are other ways to think about them which are still equivalent.
There are in fact many other models for them

- quasi-categories
- Segal categories
- complete Segal spaces

Today, we'll focus on complete Segal spaces, although there could be many other ways to finish this talk using the other models!

Segal spaces

Just as we defined simplicial sets, we have more general simplicial objects.
We're going to consider simplicial spaces, or simplicial objects in the category of simplicial sets (also called bisimplicial sets).

Definition

A (Reedy fibrant) simplicial space $W \colon \mathbf{\Delta}^{op} \to \mathcal{SS}ets$ is a *Segal space* if the Segal maps

$$\varphi_k \colon W_k \to \underbrace{W_1 \times_{W_0} \cdots \times_{W_0} W_1}_{k}$$

are weak equivalences of simplicial sets.

A Segal space should be thought of as something like a simplicial category, but with composition only up to homotopy and with a space rather than a set of objects.

Complete Segal spaces

Complete Segal spaces as $(\infty, 1)$-categories

The Θ-construction

The categories Θ_n

Θ_n-spaces

Categories enriched in Θ_n-spaces

Comparisons of models for (∞, n)-categories