Matrix factorizations over projective schemes

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Matrix factorizations

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A *matrix factorization* of f consists of

- free Q-modules G_1, G_0
- maps (i.e., matrices) $g_1 : G_1 \to G_0, \quad g_0 : G_0 \to G_1$

such that $g_1 \circ g_0 = f \cdot 1_{G_0}$ and $g_0 \circ g_1 = f \cdot 1_{G_1}$.

Example

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A matrix factorization of f is given by setting $G_1 = G_0 = Q^2$ and

$$g_1 = \begin{bmatrix} y & x \\ x^2 & -y \end{bmatrix} \quad g_0 = \begin{bmatrix} y & x \\ x^2 & -y \end{bmatrix}$$

Matrix factorizations were introduced by Eisenbud in 1980. He showed that if Q is a regular local ring, and we set $R = Q/(f)$, then the free resolution of every finitely generated R-module is determined after at most $d = \dim Q$ steps by a matrix factorization of f.

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Example: if $R = \mathbb{C}[[x,y]]/(x^3 + y^2)$, then the free resolution of $R/(x,y)$ has the form:

$$\ldots \to R^2 \xrightarrow{\begin{bmatrix} y & x \\ x^2 & -y \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} y & x \\ x^2 & -y \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} y & x \end{bmatrix}} R^1 \to 0$$

Homotopy category of matrix factorizations

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$$\underline{\mathrm{Hom}}_{MF}(\mathbb{G}, \mathbb{H}).$$

The *homotopy category of matrix factorizations*, denoted $[MF(Q, f)]$, is the homotopy category of this dg-category.
Morphisms are given by:

$$\mathrm{Hom}_{[MF]}(\mathbb{G}, \mathbb{H}) = H^0 \underline{\mathrm{Hom}}_{MF}(\mathbb{G}, \mathbb{H}).$$

To say this more concretely, if

$$\mathbb{G} = (G_1 \xrightarrow{g_1} G_0 \xrightarrow{g_0} G_1)$$

$$\mathbb{H} = (H_1 \xrightarrow{h_1} H_0 \xrightarrow{h_0} H_1)$$

are matrix factorizations of f, then an element α of $\mathrm{Hom}_{[MF]}(\mathbb{G}, \mathbb{H})$ is an equivalence class of a pair of maps

$$\alpha_1 : G_1 \to H_1 \qquad \alpha_0 : G_0 \to H_0$$

such that the following squares commute:

$$\begin{array}{ccccc} G_1 & \xrightarrow{g_1} & G_0 & \xrightarrow{g_0} & G_1 \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_1} \\ H_1 & \xrightarrow{h_1} & H_0 & \xrightarrow{h_0} & H_1, \end{array}$$

under the usual homotopy equivalence relation.

As in the case of complexes, the homotopy category of matrix factorizations is triangulated.

Motivated in part by matrix factorizations, Buchweitz introduced in 1986 the *stable or singularity category* of a ring R:

$$\mathrm{D}_{\mathrm{sg}}(R) = \mathrm{D}^b(R)/\operatorname{Perf}(R),$$

where $\mathrm{D}^b(R)$ is the bounded derived category of finitely generated R-modules and $\operatorname{Perf}(R)$ is the full subcategory of complexes quasi-isomorphic to a bounded complex of finitely generated projective R-modules, i.e., *perfect complexes*.

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Note that this category is trivial if and only if R has finite global dimension, so it in some sense captures information on the singularity of R.

If $\mathbb{G} = (G_1 \xrightarrow{g_1} G_0 \xrightarrow{g_0} G_1)$ is a matrix factorization of $f \in Q$, and $R = Q/(f)$, then $\operatorname{coker} g_1$ is naturally an R-module.
(i.e., $f \cdot G_0 \subseteq \text{Image}(g_1)$.)

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(i.e., $f \cdot G_0 \subseteq \operatorname{Image}(g_1)$.)

Buchweitz noted that in case Q is a regular local ring, the functor

$$[MF(Q, f)] \xrightarrow{\Psi} \mathrm{D_{sg}}(R)$$

which acts on objects by

$$\mathbb{G} = (G_1 \xrightarrow{g_1} G_0 \xrightarrow{g_0} G_1) \mapsto \operatorname{coker} g_1 \in \mathrm{D_{sg}}(R)$$

is a triangulated equivalence.

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The essential surjectivity of Ψ follows from Eisenbud's theorem that every R-module is determined up to syzygy by a matrix factorization and by design the syzygy functor is invertible on the category $\mathrm{D}_{\mathrm{sg}}(R)$.

The kernel of Ψ is zero because if $\operatorname{coker} g_1 = 0 \in \mathrm{D}_{\mathrm{sg}}(R)$, then $\operatorname{coker} g_1$ is a projective R-module (easy Lemma) and one can use the lifting property of projectives to find a null-homotopy of $\mathbb{G}$.

So when $R = Q/(f)$ for Q a regular local ring, we have a very concrete description of the "stable" homological algebra: it is all packaged in matrix factorizations.

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This has led to results such as Polishchuk and Vaintrob’s theory of Chern characters and Hirzebruch-Riemann-Roch theorem for modules over such a hypersurface (as discussed in Mark’s talk). These are very striking results motivated by physics that have solved some open problems in commutative algebra.

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Let X be a regular closed subscheme of projective space over some commutative Noetherian ring Q:

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Let $W \in \Gamma(X, i^*\mathcal{O}(1))$ be a nonzero global section, and let

$$Y = \mathrm{Proj}(Q[T_1, \dots, T_c]/(W)) \subseteq X$$

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Example: If $X = \mathbb{P}^0_Q \cong \operatorname{Spec} Q$, $W \in \Gamma(X, \mathcal{O}(1)) = Q$, and $R = Q/(W)$, then $Y = \operatorname{Spec} R \subseteq \operatorname{Spec} Q = X$.

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such that

$$e_0 \circ e_1 = 1 \otimes W : \mathcal{E}_1 \to \mathcal{E}_1 \otimes \mathcal{O}(1) \text{ and}$$
$$(e_1 \otimes 1_{\mathcal{O}(1)}) \circ e_0 = 1 \otimes W : \mathcal{E}_0 \to \mathcal{E}_0 \otimes \mathcal{O}(1),$$

where we think of W as a map $\mathcal{O} \to \mathcal{O}(1)$.

We are interested in these non-affine matrix factorizations because of their relation to affine complete intersection rings (for details on this see arXiv:1205.2552). Other people (Orlov, Polischuk/Vaintrob, Positselski,...) have studied non-affine matrix factorizations motivated by homological mirror symmetry.

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Our goal is to generalize the equivalence of Buchweitz by giving a relation between the "homotopy category" of non-affine matrix factorizations and the singularity category of $Y \subseteq X$ (known in the case $X = \mathbb{P}^0_Q$).

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The singularity category of a scheme Y is

$$\mathrm{D}_{\mathrm{sg}}(Y) = \mathrm{D}^b(\mathrm{coh}\, Y)/\operatorname{Perf} Y$$

where a complex of coherent sheaves on Y is perfect if it locally quasi-isomorphic to a bounded complex of free modules.

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First we naively mimic the affine case: to matrix factorizations $\mathbb{E}, \mathbb{F}$ of W, we associate a chain complex $\underline{\mathcal{H}\text{om}}_{MF}(\mathbb{E}, \mathbb{F})$ of coherent sheaves on X.

We define $[MF(X, \mathcal{O}(1), W)]_{\text{naive}}$ to be the category with morphisms

$$\text{Hom}_{[MF]_{\text{naive}}}(\mathbb{E}, \mathbb{F}) = H^0\Gamma(X, \underline{\mathcal{H}\text{om}}_{MF}(\mathbb{E}, \mathbb{F})).$$

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are matrix factorizations of W, then an element h of $\mathrm{Hom}_{[MF]_{\text{naive}}}(\mathbb{E}, \mathbb{F})$ is an equivalence class of a pair of maps

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$[MF(X, \mathcal{O}(1), W)]_{\text{naive}}$ is a triangulated category.

As in the affine case, if $\mathbb{E} = (\mathcal{E}_1 \xrightarrow{e_1} \mathcal{E}_0 \xrightarrow{e_0} \mathcal{E}_1(1))$ is a matrix factorization of W, then $\operatorname{coker} e_1$ has support contained in $Y \subseteq X$, i.e., is naturally a coherent sheaf on Y.

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There is a triangulated functor

$$\Psi : [MF(X, \mathcal{O}(1), W)]_{\text{naive}} \to \mathrm{D}_{\text{sg}}(Y)$$
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and it's not too hard to show that this functor is essentially surjective.

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Unfortunately it is not always the case that the kernel of Ψ is zero: if $\operatorname{coker} e_1 = 0 \in \mathsf{D}_{\mathsf{sg}}(Y)$, then $\operatorname{coker} e_1$ is locally free, and so $\mathbb{E}$ will locally be null-homotopic, but there are examples which show that $\mathbb{E}$ need not be null-homotopic.

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Thus Ψ will not in general be an equivalence. To find an equivalence between a category of matrix factorizations and $\mathsf{D_{sg}}(Y)$ we need to keep track of more data than is contained in $[MF(X, \mathcal{O}(1), W)]_{\mathsf{naive}}$.

Recall that the definition of morphisms in $[MF(X, \mathcal{O}(1), W)]_{\text{naive}}$ is

$$\operatorname{Hom}_{[MF]_{\text{naive}}}(\mathbb{E}, \mathbb{F}) = H^0\Gamma(X, \underline{\mathcal{H}\text{om}}_{MF}(\mathbb{E}, \mathbb{F})).$$

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To remedy the deficiency of the "naive" homotopy category, instead of taking global sections $\Gamma(X, -)$ we should take *derived global sections* or hypercohomology

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Hypercohomology may be computed using a semi-injective resolution (fibrant replacement) of $\Gamma(X, \underline{\mathcal{H}\mathrm{om}}_{MF}(\mathbb{E}, \mathbb{F}))$ or the total complex of the bicomplex which is the Cech complex on every degree of $\underline{\mathcal{H}\mathrm{om}}_{MF}(\mathbb{E}, \mathbb{F})$.

Define the *homotopy category of matrix factorizations of W* to be the category $[MF(X, \mathcal{O}(1), W)]$ with objects the same as those of $[MF(X, \mathcal{O}(1), W)]_{\text{naive}}$. Morphisms are given by

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There is a natural transformation $\Gamma(X, -) \to \mathbf{R}\Gamma(X, -)$ which gives rise to a functor

$$[MF(X, \mathcal{O}(1), W]_{\text{naive}} \to [MF(X, \mathcal{O}(1), W)]$$

that is the identity on objects.

Theorem (-, Walker)

There is a triangulated equivalence

$$\Phi : [MF(X, \mathcal{O}(1), W)] \to \mathrm{D_{sg}}(Y).$$

that fits into the following commutative diagram:

$$\begin{array}{ccc} [MF(X,\mathcal{O}(1),W)]_{naive} & \xrightarrow{\Psi} & \mathrm{D_{sg}}(Y) \\ \downarrow & \nearrow_{\Phi} & \\ [MF(X,\mathcal{O}(1),W)] & & \end{array}$$

In particular, for $\mathbb{E} = (\mathcal{E}_1 \xrightarrow{e_1} \mathcal{E}_0 \xrightarrow{e_0} \mathcal{E}_1(1)$*,*

$$\Phi(\mathbb{E}) = \mathrm{coker}\, e_1.$$

Idea of proof: the functor Φ is essentially surjective, so we need to show that it's fully faithful.

Let $\mathbb{E}, \mathbb{F}$ be objects of $[MF(X, \mathcal{O}(1), W)]$. For every $m \in \mathbb{Z}$, we have a commutative diagram

$$\begin{array}{ccc} H^m\Gamma(X, \underline{\mathcal{H}\mathrm{om}}_{MF}(\mathbb{E}, \mathbb{F})) & \longrightarrow & \mathrm{Hom}_{\mathrm{D_{sg}}(Y)}(\mathrm{coker}\, \mathbb{E}[-m], \mathrm{coker}\, \mathbb{F}) \\ \downarrow & & \\ H^m\mathbf{R}\Gamma(X, \underline{\mathcal{H}\mathrm{om}}_{MF}(\mathbb{E}, \mathbb{F})) & & \end{array}$$

The vertical arrow is an isomorphism for $m \gg 0$ by Serre Vanishing (uses that X is projective). The horizontal arrow is an isomorphism for $m \gg 0$ by classical arguments for matrix factorizations.

So for all $m \gg 0$ we have isomorphisms:

$$H^m \mathbf{R}\Gamma(X, \Gamma(X, \underline{\mathcal{H}\mathrm{om}}_{MF}(\mathbb{E}, \mathbb{F})) \to \mathrm{Hom}_{\mathrm{D_{sg}}(Y)}(\mathrm{coker}\, \mathbb{E}[-m], \mathrm{coker}\, \mathbb{F})$$

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Now we can use a totaling of the Koszul complex (also uses that X is projective) and the fact that $\mathbb{R}\Gamma$ respects quasi-isomorphisms to get an isomorphism in one degree less:

$$H^{m-1} \mathbf{R}\Gamma(X, \Gamma(X, \underline{\mathcal{H}\mathrm{om}}_{MF}(\mathbb{E}, \mathbb{F})) \to \mathrm{Hom}_{\mathrm{D_{sg}}(Y)}(\mathrm{coker}\, \mathbb{E}[-m+1], \mathrm{coker}\, \mathbb{F})$$

and by induction we have an isomorphism for $m = 0$.

Polishchuk and Vaintrob proved a similar theorem (which was posted before ours, works for smooth Deligne-Mumford stacks, and does not require X to be projective). They showed that the Verdier quotient of $[MF]_{\text{naive}}$ by the locally nullhomotopic objects is equivalent to $\mathrm{D}_{\text{sg}}(Y)$.

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Our version has two advantages:

- The Hom-spaces in our category have a very explicit description in terms of a Cech complex which gives rise to a useful spectral sequence.

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Our version has two advantages:

- The Hom-spaces in our category have a very explicit description in terms of a Cech complex which gives rise to a useful spectral sequence.
- Our methods let us prove a relative version in which X is not assumed to be regular.

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