Homotopy fiber products of homotopy theories

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Homotopy pullbacks in a model category

Suppose we have a model category $\mathcal{M}$ and objects A, B, C fitting into a diagram

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C. \end{tikzcd}$$

We can define the ordinary pullback $A \times_C B$, but it is not necessarily homotopy invariant.

In other words, replacing one or more of these objects with weakly equivalent ones will not necessarily result in a weakly equivalent pullback.

We can fix this problem by taking the homotopy pullback $A \times_C^h B$, in which we replace at least one of the maps f and g with a fibration before taking the pullback.

This construction is homotopy invariant, and it does not depend on which map we replace.

Thus, homotopically speaking, the homotopy pullback is the correct construction to make.

What if the diagram we want to consider does not live in a model category?

In particular, what if we wanted the objects in the diagram to be model categories? There is no known “model category of model categories” so this standard construction cannot be used.

Can we find an alternative definition?

Fiber products of model categories (Smith, Toën)

Consider a diagram of left Quillen functors of model categories

$$\begin{tikzcd} & \mathcal{M}_2 \arrow[d, "F_2"] \\ \mathcal{M}_1 \arrow[r, "F_1"] & \mathcal{M}_3 \end{tikzcd}$$

We want to have a sensible notion of homotopy pullback

$$\mathcal{M} = \mathcal{M}_1 \times^h_{\mathcal{M}_3} \mathcal{M}_2$$

for this diagram.

Let $\mathcal{M}$ have as objects 5-tuples

$$(x_1, x_2, x_3; u, v)$$

where each x_i is an object of $\mathcal{M}_i$ and u and v are maps

$$F_1(x_1) \xrightarrow{u} x_3 \xleftarrow{v} F_2(x_2).$$

The morphisms in $\mathcal{M}$ are given by

$$(f_1, f_2, f_3)\colon (x_1, x_2, x_3; u, v) \to (y_1, y_2, y_3; u', v')$$

where each

$$f_i\colon x_i \to y_i$$

is a morphism of $\mathcal{M}_i$ and the following diagram commutes:

$$\begin{tikzcd}
F_1(x_1) \arrow[r, "u"] \arrow[d, "F_1(f_1)"'] & x_3 \arrow[d, "f_3"] & F_2(x_2) \arrow[l, "v"'] \arrow[d, "F_2(f_2)"] \\
F_1(y_1) \arrow[r, "u'"] & y_3 & F_2(y_2). \arrow[l, "v'"']
\end{tikzcd}$$

There is a natural model category structure on $\mathcal{M}$.

- The weak equivalences are the triples (f_1, f_2, f_3) for which each

  $$f_i \colon x_i \to y_i$$

  is a weak equivalence in $\mathcal{M}_i$.
- The cofibrations are the triples (f_1, f_2, f_3) for which each f_i is a cofibration in $\mathcal{M}_i$.

We really want to require that the maps u and v be weak equivalences, but with this additional restriction, $\mathcal{M}$ will no longer be a model category, since it will longer have enough limits and colimits.

Presumably, we can find a localization of the model category $\mathcal{M}$ so that the fibrant and cofibrant objects have u and v weak equivalences.

Alternatively, we could just require u and v to be weak equivalences and not worry about the model structure.

Either way, we will assume from now on that $\mathcal{M}$ has this additional condition imposed.

- Is this really the correct notion of "homotopy fiber product"?
- How could we tell?

We would like to consider model categories as objects in some model category so that we can determine whether these fiber products really correspond to homotopy pullbacks.

The homotopy theory of homotopy theories

There is no model category of model categories, but if we consider a more general notion of homotopy theory, there is a homotopy theory of homotopy theories.

We'll consider a "homotopy theory" to be a category with weak equivalences.

Via Dwyer and Kan's simplicial localization techniques, we can consider simplicial categories and homotopy theories to be the same thing.

- Every category with weak equivalences gives rise to a simplicial category.
- Up to equivalence, every simplicial category can be obtained in this way.

Models for homotopy theories

There are, however, three other possible ways to regard homotopy theories as mathematical objects.

- Simplicial categories
- Complete Segal spaces
- Segal categories
- Quasi-categories

There is an appropriate model category structure corresponding to each, and they are all Quillen equivalent.

Here, complete Segal spaces are the preferred models because weak equivalences are easiest to identify in this setting.

Complete Segal spaces

Complete Segal spaces are simplicial spaces W satisfying the Segal condition

$$W_n \simeq \underbrace{W_1 \times_{W_0} \cdots \times_{W_0} W_1}_{n}$$

and an additional condition saying that W_0 is equivalent to the "homotopy equivalences" in W_1.

Theorem (Rezk)

- *There is a model category structure $\mathcal{CSS}$ on the category of simplicial spaces such that the fibrant and cofibrant objects are complete Segal spaces.*
- *The weak equivalences between complete Segal spaces are levelwise weak equivalences of simplicial sets.*

There is a natural functor taking a model category to a complete Segal space, essentially given by

$$W_n = \mathrm{nerve}(\mathrm{we}(\mathcal{M}^{[n]}))$$

where $\mathrm{we}(\mathcal{M}^{[n]})$ has objects n-chains of composable morphisms in $\mathcal{M}$

$$m_0 \to m_1 \to \cdots \to m_n$$

and morphisms

$$\begin{tikzcd} m_0 \arrow[r] \arrow[d, "\simeq"] & m_1 \arrow[r] \arrow[d, "\simeq"] & \cdots \arrow[r] & m_n \arrow[d, "\simeq"] \\ m_0' \arrow[r] & m_1' \arrow[r] & \cdots \arrow[r] & m_n' \end{tikzcd}$$

We denote this functor L_C.

Theorem (B)

Given a model category $\mathcal{M}$, its image under L_C looks like

$$\coprod_{\langle x \rangle} BAut^h(x) \Longleftarrow \coprod_{\langle \alpha \colon x \to y \rangle} BAut^h(\alpha) \Lleftarrow \cdots$$

where $\langle x \rangle$ denotes the weak equivalence class of x in $\mathcal{M}$, $\langle \alpha \rangle$ denotes the weak equivalence class of α in $\mathcal{M}^{[1]}$, and $Aut^h(x)$ denotes the monoid of self weak equivalences of x.

Using this theorem, if we have a diagram of model categories

$$\begin{tikzcd} & \mathcal{M}_2 \arrow[d, "F_2"] \\ \mathcal{M}_1 \arrow[r, "F_1"] & \mathcal{M}_3, \end{tikzcd}$$

we can compare the homotopy types of $L_C(\mathcal{M})$ and the homotopy pullback of the diagram of complete Segal spaces

$$\begin{tikzcd} & L_C(\mathcal{M}_2) \arrow[d, "L_C(F_2)"] \\ L_C(\mathcal{M}_1) \arrow[r, "L_C(F_1)"] & L_C(\mathcal{M}_3). \end{tikzcd}$$

Theorem (B)

There is a weak equivalence

$$L_C(\mathcal{M}) \simeq L_C(\mathcal{M}_1) \times^h_{L_C(\mathcal{M}_3)} L_C(\mathcal{M}_2).$$

In other words, taking the functor L_C commutes with taking the homotopy fiber product, either in the sense of model categories or in the usual sense for complete Segal spaces.

Thus, when we require the maps u and v to be weak equivalences, this notion of fiber product of model categories is the correct one.

Potential applications

- Generalizing Toën's derived Hall algebras to more general stable homotopy theories
- Studying recollements for stable homotopy theories rather than for derived categories or stable model categories.
- Better understanding sheaf-type conditions for homotopy theories.

References

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