Configuration spaces and operad actions

Joint with K. Hess

Configurations

Definition $C_k X = \{k\text{-tuples of distinct points in } X\}$

- $C_* X$ a symmetric sequence
- study its homotopy type

Properties

- functorial for embeddings
- $C_*(X \cup Y) \cong C_* X \boxtimes C_* Y$ (exponential)

Basic question

- $C_*(X \times Y) \cong ??$ (super exponential??)

Points fuzzed up to disks

$$C_k M \sim \mathrm{Emb}^{\mathrm{ts}}(k \cdot D^m, M)$$

Little m-disk operad $E(m)$

$$E(m)_k \sim \mathrm{Emb}^{\mathrm{ts}}(k \cdot D^m, D^m)$$

Composition action

$C_* M$ a right module over $E(m)$

Boardman-Vogt operad $\otimes$ product F-V,L

$$E(m) \otimes^{\mathrm{h}} E(n) \sim E(n+m)$$

External right module $\square$ product $\mathcal{P}$, $\mathcal{Q}$ operads

$$A_{\mathcal{P}} \square B_{\mathcal{Q}} \;\in\; \mathrm{Mod}_{\mathcal{P}\otimes\mathcal{Q}}$$

Main theorem

$$C_*(M \times N) \sim C_*M \square^{\mathrm{h}} C_*N \qquad \text{in } \mathrm{Mod}_{E(m+n)}$$

Determining $\Box$

$\Box$ for free modules (Superexponential)

$$\text{Free}_{\mathcal{P}}(x_i) \Box \text{Free}_{\mathcal{Q}}(x_j) \cong \text{Free}_{\mathcal{P}\otimes\mathcal{Q}}(x_{ij})$$

$\Box$ in general (Lurking adjointness)

$A \Box B$ commutes with colimits in A and B

Example

$$Free_{E(m)}(x_i) \sim C_*(i \cdot D^m)$$

$$C_*(i \cdot D^m) \Box C_*(j \cdot D^n) \sim C_*(i \cdot j \cdot D^{m+n})$$

Patching configurations (Get ready for proof)

$C_*(X \times \mathbb{R})$ a graded A_∞ monoid Juxtaposition

$$2 \cdot (X \times \mathbb{R}) \quad \hookrightarrow \quad X \times \mathbb{R}$$

$$C_*(X \times \mathbb{R})^{\boxtimes 2} \to C_*(X \times \mathbb{R})$$

The monoid acts $X \subset M$, codim 1, separating

$$M = A \cup_{X \times \mathbb{R}} B$$

$$A \cup (X \times \mathbb{R}) \hookrightarrow A \qquad (X \times \mathbb{R}) \cup B \hookrightarrow B$$

$C_*(X \times \mathbb{R})$ acts on C_*A, C_*B

Mayer-Vietoris (J. Francis)

$$C_*M \sim C_*A \boxtimes^{h}_{C_*(X \times \mathbb{R})} C_*B$$

Proof by slicing

Theorem: $C_*(M \times N) \sim C_*M \square C_*N$

- OK if $M = i \cdot D^m$ and $N = j \cdot D^n$ (example)
- OK if $M = i \cdot D^m$ (slice N to pieces, use MV)
- OK in general (slice M to pieces, use MV)