Completeness of the Gorenstein projective cotorsion pair.

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Gorenstein projective modules

Let R be any ring.

R-Mod $=$ Category of R-modules.

$\mathrm{Ch}(R)$ $=$ Category of chain complexes of R-modules.

Definition: An R-module M is called **Gorenstein projective** if there exists an exact complex of projectives

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

with $M = \ker(P^0 \to P^1)$ and which remains exact after applying $\mathrm{Hom}_R(-, P)$ for any projective module P.

Gorenstein projective modules

Set $\mathcal{GP}$ = Class of all Gorenstein projective modules.

Set $\mathcal{V} = \mathcal{GP}^{\perp} = \{V \in R\text{-Mod} \,|\, \mathrm{Ext}^1_R(M, V) = 0 \ \ \forall\, M \in \mathcal{GP}\}$.

Question: For what rings R is $(\mathcal{GP}, \mathcal{V})$ a complete cotorsion pair?

- To be a *cotorsion pair* means $\mathcal{V} = \mathcal{GP}^{\perp}$ and $\mathcal{GP} = {}^{\perp}\mathcal{V}$.
- To be a *complete cotorsion pair* means that in addition the following condition (and its dual) holds: For any R-module N there exists a short exact sequence

$$0 \to V \to M \to N \to 0 \text{ with } M \in \mathcal{GP} \text{ and } V \in \mathcal{V}.$$

Why is this interesting?

Proposition: If $(\mathcal{GP}, \mathcal{V})$ is a complete cotorsion pair then there is a Quillen model structure on R-Mod with:

1. Cofibrant objects $= \mathcal{GP}$.
2. Fibrant objects $=$ All R-modules.
3. Trivial objects $= \mathcal{V} = \mathcal{GP}^{\perp}$.
4. Trivially cofibrant objects $=$ All projective modules.

Illustration: The stable module category of R

Special Case: Assume R is quasi-Frobenius.

Then $(\mathcal{GP}, \mathcal{V}) =$ (All modules, Projective-injective modules).

The corresponding model structure on R-Mod has

$$\text{Ho}(R\text{-Mod}) \;=\; R\text{-Mod}/\sim$$

where $f \sim g$ iff $f - g$ factors through a projective-injective module.

Generalization (Hovey 2001): If R is a Gorenstein ring then $(\mathcal{GP}, \mathcal{V})$ is a complete cotorsion pair.

Totally acyclic complexes of projectives

IDEA: Gorenstein projective modules are the 0-cycles of certain complexes. Focus on those complexes instead.

Definition: A chain complex of projectives

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

is called a **totally acyclic complex of projectives** if it is exact (acyclic) and remains exact after applying $\mathrm{Hom}_R(-, P)$ for any projective module P.

Constructing projective model structures on $\mathsf{Ch}(R)$

Main Theorem: Suppose we have the following setup in $\mathsf{Ch}(R)$:

- $\mathcal{C}$ is some class of chain complexes of projectives.
- $\mathcal{W} = \mathcal{C}^{\perp} = \{\, W \in \mathsf{Ch}(R) \mid \mathsf{Ext}^1_{\mathsf{Ch}(R)}(C, W) = 0 \ \ \forall\, C \in \mathcal{C} \,\}$.
- There is a "**test module**" A satisfying the following:
  A complex C of projectives is in $\mathcal{C}$ iff $A \otimes_R C$ is exact.

Then there is a cofibrantly generated abelian model structure on $\mathsf{Ch}(R)$ described as follows:

1. Cofibrant objects $= \mathcal{C}$.
2. Fibrant objects $=$ All complexes.
3. Trivial objects $= \mathcal{W} = \mathcal{C}^{\perp}$.
4. Trivially cofibrant objects $=$ All projective complexes.

Corollary 1: In the Main Theorem...

- Take $\mathcal{C}$ to be the class of ALL complexes of projectives.
- Then $A = 0$ serves as a "test module". That is,
  A complex of projectives C is in $\mathcal{C}$ iff $0 \otimes_R C$ is exact.

We call the corresponding model structure on Ch(R) the *Proj model structure*. It is cofibrantly generated and satisfies:

1. Cofibrant objects $= \mathcal{C} =$ All complexes of projectives.
2. Fibrant objects $=$ All complexes.
3. Trivial objects $= \mathcal{W} = \mathcal{C}^{\perp}$
4. Trivially cofibrant objects $=$ All projective complexes.

Corollary 2: In the Main Theorem...

- Take $\mathcal{C}$ to be the class of all exact complexes of projectives.
- Then $A = R$ serves as a "test module". That is,
  A complex of projectives C is in $\mathcal{C}$ iff $R \otimes_R C$ is exact.

We call the corresponding model structure on $\mathrm{Ch}(R)$ the *exact Proj model structure*. It is cofibrantly generated and satisfies:

1. Cofibrant objects $= \mathcal{C} =$ All exact complexes of projectives.
2. Fibrant objects $=$ All complexes.
3. Trivial objects $= \mathcal{W} = \mathcal{C}^{\perp}$
4. Trivially cofibrant objects $=$ All projective complexes.

How to construct a "totally acyclic Proj model structure"?

Question: Is there a "test module" A for the class $\mathcal{C}$ of totally acyclic complexes of projectives so that we get a *"totally acyclic Proj model structure"* on $\mathrm{Ch}(R)$?

Answer: Yes, but...

1. Need R to be a coherent ring and to satisfy the condition that all flat modules have finite projective dimension.
2. The solution points to a similar model structure that exists for ANY coherent ring!
   - Strengthen "totally acyclic" to "steadfastly acyclic" complexes.
   - Analog: "Gorenstein projective" to "Ding projective" modules.

Ding projective modules

- A **steadfastly acyclic complex of projectives** is an exact sequence of projective modules

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

  which remains exact after applying $\mathrm{Hom}_R(-, F)$ for any flat module F.

- We call an R-module M **Ding projective** if there exists a steadfastly acyclic complex of projectives

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

  with $M = \ker(P^0 \to P^1)$.

Equivalence of Ding projective and Gorenstein projective

Proposition: Let R be any ring satisfying the condition that all flat modules have finite projective dimension. Then

1. A complex of projectives is totally acyclic if and only if it is steadfastly acyclic.
2. An R-module is Gorenstein projective iff it is Ding projective.

Corollary 3: Assume R is coherent. In the Main Theorem...

- Take $\mathcal{C}$ = class of steadfastly acyclic complexes of projectives.
- Let $\kappa > |R|$ be a regular cardinal and $\{E_\alpha\}_{\alpha \in I}$ be the "set" of all FP-injectives modules with $|E_\alpha| \leq \kappa$.

Lemmas: $A = R \oplus (\oplus_{\alpha \in I} E_\alpha)$ serves as a "test module" for $\mathcal{C}$.

So there is a projective model structure on Ch(R) having $\mathcal{C}$ as the class of cofibrant objects. We call this the **steadfastly acyclic Proj model structure** on Ch(R).

But if R satisfies the condition that all flat modules have finite projective dimension then we may call it the **totally acyclic Proj model structure** on Ch(R).

Corollary 3' (Module version of Corollary 3): R any coherent ring. Let $\mathcal{DP}$ = The class of all Ding projective R-modules. Then $(\mathcal{DP}, \mathcal{DP}^{\perp})$ is a complete cotorsion pair and gives rise to a cofibrantly generated abelian model structure on R-Mod. We call this the **Ding projective model structure** on R-Mod.

If R satisfies the condition that all flat modules have finite projective dimension then $\mathcal{DP} = \mathcal{GP}$ and we call it the **Gorenstein projective model structure** on R-Mod.

FACT: The Ding projective model structure on R-Mod is Quillen equivalent to the steadfastly acyclic Proj model on Ch(R).

Examples of these rings

There is an abundance of rings having the property that all flat modules have finite projective dimension!

Examples: Let R be any ring.

1. (Simson 1974) If $|R| \leq \aleph_n$ then $\mathrm{pd}(F) \leq n+1$ for all flat F.
2. Enochs, Jenda and López-Ramos have studied *n-perfect* rings: $\mathrm{pd}(F) \leq n$ for all flat F.
   - A perfect ring is 0-perfect.
   - An n-Gorenstein ring is n-perfect.
3. (Jorgensen 2005) Any Noetherian ring of finite Krull dimension has the property that all flat modules have finite projective dimension.

Thank You!