Homotopy theories in topology and algebra

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Classical homotopy theory

Let X and Y be topological spaces, each with a specified basepoint.

General theme in algebraic topology:

Assign to a space X an algebraic object (a group or ring) and to any map of spaces $X \to Y$ an appropriate homomorphism.

Definition

The *fundamental group* of a pointed space X is

$$\pi_1(X) = [S^1, X]_*,$$

the group of homotopy classes of maps $S^1 \to X$ which preserve basepoints.

More generally, we have the following definition.

Definition

For any $n \geq 0$, we can define the nth homotopy group

$$\pi_n(X) = [S^n, X]_*.$$

For $n = 0$, this is just the set of path components of X.

For $n \geq 1$, $\pi_n(X)$ is a group, and for $n \geq 2$, $\pi_n(X)$ is an abelian group.

Homotopy groups are also *functorial* in that, given any map $f\colon X \to Y$, we get a map

$$\pi_n(f)\colon \pi_n(X) \to \pi_n(Y)$$

which is a group homomorphism when $n \geq 1$.

Definition

A map $f\colon X \to Y$ is a *weak homotopy equivalence* if the map

$$\pi_n(f)\colon \pi_n(X) \to \pi_n(Y)$$

is an isomorphism for all n.

Definition

A map $f\colon X \to Y$ is a *homotopy equivalence* if there exists a map $g\colon Y \to X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$.

Every homotopy equivalence is a weak homotopy equivalence, but not conversely.

Theorem (Whitehead)

If X and Y are CW complexes, then a weak homotopy equivalence $f\colon X \to Y$ is a homotopy equivalence.

Our goal is to study spaces up to weak homotopy equivalence.

In view of Whitehead's Theorem, we can reduce to considering CW complexes, by the following theorem.

Theorem (CW Approximation)

For any space Z, there exists a CW complex X together with a weak homotopy equivalence $X \to Z$.

Then, we can take homotopy classes of maps. Since homotopy equivalences have inverses up to homotopy, their homotopy classes have inverses on the nose.

Thus, we have formally inverted the weak homotopy equivalences in a nice way.

The category of topological spaces (or CW complexes) with homotopy classes of maps between them is called the *homotopy category of spaces*.

We now turn to a similar situation in algebra.

Homological algebra

Consider non-negatively graded chain complexes of R-modules for a fixed ring R. Such a chain complex M_* looks like

$$0 \xleftarrow{\partial_0} M_0 \xleftarrow{\partial_1} M_1 \xleftarrow{\partial_2} M_2 \xleftarrow{\partial_3} \cdots .$$

By the definition of chain complex, we have

$$\partial_n \circ \partial_{n+1} = 0.$$

In other words,

$$\mathrm{im}(\partial_{n+1}) \subseteq \ker(\partial_n).$$

Definition

The *nth homology group* of M_* is defined to be

$$H_n(M_*) = \frac{\ker(\partial_n)}{\operatorname{im}(\partial_{n+1})}.$$

A map of chain complexes $f\colon M_* \to N_*$ is a collection of maps $f_n\colon M_n \to N_n$ making the diagram

$$\begin{tikzcd}
0 & M_0 \arrow[l, "\partial_0"'] \arrow[d, "f_0"] & M_1 \arrow[l, "\partial_1"'] \arrow[d, "f_1"] & M_2 \arrow[l, "\partial_2"'] \arrow[d, "f_2"] & \cdots \arrow[l, "\partial_3"'] \\
0 & N_0 \arrow[l, "\partial_0'"'] & N_1 \arrow[l, "\partial_1'"'] & N_2 \arrow[l, "\partial_2'"'] & \cdots \arrow[l, "\partial_3'"']
\end{tikzcd}$$

commute.

Such a map f induces a map $H_n(f)\colon H_n(M_*) \to H_n(N_*)$.

Definition

A map $f_*\colon M_* \to N_*$ is a *homology equivalence* if the map

$$H_n(f)\colon H_n(M_*) \to H_n(N_*)$$

is an isomorphism for all $n \geq 0$.

These homology equivalences play the same role for chain complexes that weak homotopy equivalences do for spaces, and we would like to formally invert them.

For chain complexes, the “nice” objects that we can replace with are the projective ones, or chain complexes made up of projective modules.

If we replace with projectives and take homotopy classes of maps, we get the derived category, which is the analogue of the homotopy category in topology.

Can we do this same kind of thing more generally?

Abstract homotopy theory

Suppose we have a category of some kind of mathematical objects and appropriate maps between them.

To discuss a homotopy theory in this setting, we first need some notion of weak equivalence. These maps must satisfy

- the 2-out-of-3 property: if f and g are weak equivalences such that gf is defined, then if any two of these maps is a weak equivalence, then so is the third, and
- closed under retracts.

These are the maps we would like to formally invert (assuming they are not all isomorphisms).

We could just formally invert the weak equivalences, by adding in inverses to them and taking all possible composites of maps to get a category.

However, in doing so, we could get into some set-theoretic problems, in that we might be adding a proper class of maps.

We can instead provide the additional structure of a model category, which enables you to define “nice” objects and a notion of “homotopy classes of maps.”

Simplicial sets and homotopy theories

To give another example of a homotopy theory, we consider simplicial sets, which are combinatorial models for spaces.

We begin with directed simplicial complexes, which can either be drawn as spaces, or written as a diagram of sets with face maps between them:

$$\{0-\text{simplices}\} \Leftarrow \{1-\text{simplices}\} \Lleftarrow \cdots .$$

We want to generalize simplicial complexes in two ways:

- Consider 0-simplices as "degenerate" 1-simplices, etc., and
- Allow non-triangles.

Thus, in addition to face maps, we get degeneracy maps as well

$$\{0-\text{simplices}\} \to \{1-\text{simplices}\} \Rrightarrow \cdots .$$

Just as with simplicial complexes, we can geometrically realize simplicial sets as topological spaces.

Definition

A map $f : X \to Y$ of simplicial sets is a *weak equivalence* if the induced map after geometric realization is a weak homotopy equivalence of spaces.

Theorem (Quillen)

The homotopy theory of spaces is equivalent to the homotopy theory of simplicial sets.

Thus, from the viewpoint of homotopy theory, we often think of simplicial sets as “spaces.”

But, what is a homotopy theory, anyway?

It could mean a model category, and that is how Quillen proved this theorem.

However, it is often hard to show that a model category exists, so often we just want to think of a category with weak equivalences as a homotopy theory.

It would be nice to find a more concrete mathematical object that we could justifiably call a homotopy theory.

Definition

A *simplicial category* is a category with a set of objects and a simplicial set of morphisms between any two objects.

There is a natural notion of weak equivalence between simplicial categories, called a *Dwyer-Kan* or simply *DK-equivalence*.

Theorem

(Dwyer-Kan) Any category with weak equivalences gives rise to a simplicial category, and up to DK-equivalence, any simplicial category arises in this way.

Thus, a simplicial category is a model for a homotopy theory.

But, notice that we have weak equivalences of simplicial categories, and so we have a "homotopy theory of homotopy theories."

Theorem

There is a model category structure on the category of (small) simplicial categories.

Thus, we can talk about the homotopy theory of homotopy theories precisely.

However, simplicial categories are difficult to work with, and the weak equivalences are difficult to identify, so we'd like to find a nicer model.

Complete Segal spaces as homotopy theories

To come up with another way to think about homotopy theories, we begin with a standard construction for obtaining a space from a category.

Associated to a category $\mathcal{C}$ is a space called its *nerve*.

- For every object of $\mathcal{C}$, we take a 0-simplex.
- For every morphism of $\mathcal{C}$, we take a 1-simplex.
- For every pair of composable morphisms, we take a 2-simplex, etc.

The problem with the nerve is that we can have two categories that are not equivalent but whose nerves are weakly equivalent.

This difficulty arises because the nerve construction does not distinguish an isomorphism from any other morphism.

We can modify the nerve construction to get a *simplicial space*, or simplicial diagram of spaces.

We use the following steps to get our modified nerve:

- The 0th space is the nerve of the subcategory of isomorphisms of $\mathcal{C}$, called $\mathrm{iso}(\mathcal{C})$.
- The 1st space is the nerve of the isomorphisms in the morphism category of $\mathcal{C}$, called $\mathrm{iso}(\mathcal{C}^{[1]})$.
- The nth space is the nerve of $\mathrm{iso}(\mathcal{C}^{[n]})$.

This *classifying diagram*, which we call $N\mathcal{C}$, satisfies two important properties:

- $(N\mathcal{C})_n \simeq (N\mathcal{C})_1 \times_{(N\mathcal{C})_0} \cdots \times_{(N\mathcal{C})_0} (N\mathcal{C})_1$, making it look like it has composable maps, and
- thinking of $(N\mathcal{C})_1$ as the space of maps, we can take the subspace of "homotopy equivalences;" this space is weakly equivalent to $(N\mathcal{C})_0$.

A simplicial space satisfying these two conditions is called a *complete Segal space*.

Theorem (Rezk)

There is a model category structure on the category of simplicial spaces in which the "nice" objects are the complete Segal spaces. The weak equivalences between complete Segal spaces are just levelwise weak equivalences of spaces.

Theorem

The homotopy theory of simplicial categories is equivalent to the homotopy theory of complete Segal spaces.

Theorem

The complete Segal space associated to a simplicial category can be characterized up to weak equivalence.

The expected applications of these results include:

- derived Hall algebras in representation theory
- Floer homotopy theory