DG homological algebra and solution to a question of Vasconcelos

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Notation

$\mathfrak{S}(R) = \{$*isomorphism classes of semidualizing R-modules*$\}$.

Fact (Base-change)

If $R \to S$ is a local homomorphism of finite flat dimension, then $\mathfrak{S}(R) \hookrightarrow \mathfrak{S}(S)$ by $C \mapsto S \otimes_R C$.

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Outline of proof

There is a flat local ring homomorphism $R \to (R', \mathfrak{m}R', \overline{k})$. Let $\mathbf{x} \in \mathfrak{m}R'$ be a maximal R'-sequence. Then $R'/\mathbf{x}R'$ is artinian and $\mathfrak{S}(R) \hookrightarrow \mathfrak{S}(R') \hookrightarrow \mathfrak{S}(R'/\mathbf{x}R')$. A result of Happel shows that $\mathfrak{S}(R'/\mathbf{x}R')$ is finite.

Definition

A commutative differential graded (DG) R-algebra is

1. a graded commutative R-algebra $A = \bigoplus_{i=0}^{\infty} A_i$ with
2. a differential ∂^A (i.e., a sequence of R-linear maps $\partial_i^A \colon A_i \to A_{i-1}$ such that $\partial_{i-1}^A \partial_i^A = 0$ for all i) such that ∂^A satisfies the Leibniz Rule: for all $a_i \in A_i$ and $a_j \in A_j$

$$\partial_{i+j}^A(a_i a_j) = \partial_i^A(a_i)a_j + (-1)^i a_i \partial_j^A(a_j).$$

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Example (The Koszul complex)

$K = K^R(\mathbf{x})$ is a DG R-algebra for each sequence $\mathbf{x} \in R$.

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2. Let $\mathbf{x} \in \mathfrak{m}R'$ be minimal generating sequence and $K = K^{R'}(\mathbf{x})$. Now, there exists a finite dimensional DG algebra U over $\overline{k}$ such that

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3. This diagram and the lifting result imply that

$$\mathfrak{S}_{\mathrm{dg}}(R) \hookrightarrow \mathfrak{S}_{\mathrm{dg}}(R') \simeq \mathfrak{S}_{\mathrm{dg}}(K) \simeq \mathfrak{S}_{\mathrm{dg}}(U).$$

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7. We prove that there can only be finitely many open orbits, so $\mathfrak{S}_{\mathrm{dg}}(U)$ is finite.