Exceptional Lie Groups, Commutators, and Commutative Homology Rings.

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Assumptions

Unless specified,

- All topological spaces:
  - have the homotopy type of a CW complex of finite type
  - possess a basepoint.
- The symbol X will denote a finite simply-connected space.
- All maps between spaces are continuous and respect the basepoint.
- When talking about Lie groups, the symbol μ denotes the Lie group multiplication.

Assumptions

- Algebras have $\mathbb{F}_p$ (p a fixed prime) as their base field.
- All algebras and rings will be associative, graded, and finitely generated in each degree.
- A homomorphism between algebras means a graded algebra homomorphism.
- The homomorphism $T^* : A \otimes A \to A \otimes A$ is given by

$$T^*(a \otimes b) = (-1)^{|a||b|} b \otimes a$$

Commutators

Definition

The commutator map $com : G \times G \to G$ of a Lie group G is

$$com(g, h) = ghg^{-1}h^{-1}.$$

Recall: A compact connected Lie group G is abelian iff com is nullhomotopic iff it is a torus.

Question in Homology

The Lie group multiplication map $\mu : G \times G \to G$ induces an algebra structure in homology:

$$\mu_* : H_*(G;\mathbb{F}_p) \otimes H_*(G;\mathbb{F}_p) \to H_*(G;\mathbb{F}_p)$$

The homology $H_*(G;\mathbb{F}_p) = H_*(G,\mu;\mathbb{F}_p)$ is an associative algebra with multiplication μ.

Question in Homology

- If *com* is nullhomotopic, $H_*(G,\mu;\mathbb{F}_p)$ is (graded) commutative.
- Is the converse true?
- Naive conjecture: G is abelian iff $H_*(G,\mu;\mathbb{F}_p)$ is commutative.

Examples

- If G is any torus:
    - *com* is nullhomotopic and $H_*(G,\mu;\mathbb{F}_3)$ is commutative
- If $G = F_4$:
    - *com* is not nullhomotopic
      and $H_*(G,\mu;\mathbb{F}_3)$ is not commutative
- If $G = Sp(4)$:
    - *com* is not nullhomotopic,
      but $H_*(G,\mu;\mathbb{F}_3)$ is commutative

First Goal

- Look at the induced homomorphism of *com* on cohomology:

$$com^* : H^*(G,\mu;\mathbb{F}_p) \to H^*(G,\mu;\mathbb{F}_p) \otimes H^*(G,\mu;\mathbb{F}_p)$$

- Find connection between commutators in homology and com^*.

Other Multiplication Maps on G

- If $G = F_4, E_6, E_7, E_8$: $H_*(G, \mu; \mathbb{F}_3)$ is not commutative.
- Is there another binary operation $\nu : G \times G \to G$ which induces a commutative algebra structure on $H_*(G; \mathbb{F}_3) = H_*(G, \nu; \mathbb{F}_3)$?
- What kind of space is (G, ν)?

Outline

- Homotopy-Associative H-spaces
- Homology and Cohomology over $\mathbb{F}_p$
- New Multiplication Maps on a Homotopy-Associative H-space
- Open Questions

H-spaces

- Let X be a topological space with basepoint x_0. (X, μ) is an H-space if these are homotopic:

$$x \mapsto \mu(x, x_0)$$

$$x \mapsto \mu(x_0, x)$$

$$x \mapsto x$$

Homotopy-Associative H-spaces

- (X, μ) is a homotopy-associative H-space (HA-space) if these are homotopic:

$$(x, y, z) \mapsto \mu(x, \mu(y, z)) = x(yz)$$

$$(x, y, z) \mapsto \mu(\mu(x, y), z) = (xy)z$$

Homotopy Inverse Maps

- In addition, HA-spaces (X, μ) also have a (two sided) homotopy inverse map $i : X \to X$ such that these are homotopic:

$$x \mapsto \mu(x, i(x))$$

$$x \mapsto \mu(i(x), x)$$

$$x \mapsto x_0$$

Commutator on an HA-space

Definition

Let (X, μ) be a finite simply-connected HA-space with homotopy inverse i. We define the commutator $com : X \times X \to X$ as

$$com(x,y) = \mu\left(\mu(x,y), i(\mu(y,x))\right).$$

Cohomology and Homology

- $H^*(X;\mathbb{F}_p) = H^*(X,\mu;\mathbb{F}_p)$ is a Hopf algebra with product Δ^* and coproduct μ^*.
    - Reduced coproduct: $\overline{\mu}^*(x) = \mu^*(x) - x \otimes 1 - 1 \otimes x$
- $H_*(X;\mathbb{F}_p) = H_*(X,\mu;\mathbb{F}_p)$ is a Hopf algebra with product μ_*.

Commutator in Homology

Definition

The commutator $[\bar{x}, \bar{y}]$ in $H_*(X, \mu; \mathbb{F}_p)$ is given by

$$[\bar{x}, \bar{y}] = \mu_*(\bar{x}, \bar{y}) - (-1)^{|\bar{x}||\bar{y}|}\mu_*(\bar{y}, \bar{x}).$$

Cocommutator in Cohomology

Definition

The cocommutator of an element $x \in H^*(X,\mu;\mathbb{F}_p)$ is given by the map

$$\overline{\mu}^* - T^*\overline{\mu}^* : H^*(X,\mu;\mathbb{F}_p) \to H^*(X,\mu;\mathbb{F}_p) \otimes H^*(X,\mu;\mathbb{F}_p)$$

Induced Homomorphism of *com*

$com^* : H^*(X,\mu;\mathbb{F}_p) \to H^*(X,\mu;\mathbb{F}_p) \otimes H^*(X,\mu;\mathbb{F}_p)$ given by

$$com^* = \Delta^*(\mu^* \otimes (T^*\mu^*))(1 \otimes i^*)\mu^*$$

Commutative Homology

$$H^*(\mathrm{Sp}(4), \mu; \mathbb{F}_3) \cong \wedge(x_3, x_7, x_{11}, x_{15})$$

Choose x_i so that $\bar{\mu}^*(x_i) = 0$

$H_*(\mathrm{Sp}(4), \mu; \mathbb{F}_3)$ is commutative

Commutative Homology

$$\bar{\mu}^*(x_i) = 0 :$$

$$(\overline{\mu}^* - T^*\overline{\mu}^*)(x_i) = 0$$

$$com^*(x_i) = 0$$

A Nontrivial Commutator

$$H^*(F_4,\mu;\mathbb{F}_3) \cong \wedge(x_3,x_7,x_{11},x_{15}) \otimes \mathbb{F}_3[x_8]/(x_8^3)$$

$$\bar{\mu}^*(x_{11}) = x_8 \otimes x_3$$

$$(\overline{\mu}^* - T^*\overline{\mu}^*)(x_{11}) = x_8 \otimes x_3 - x_3 \otimes x_8,$$

$$[\overline{x_8},\overline{x_3}] \neq 0 \text{ in } H_*(F_4,\mu;\mathbb{F}_3)$$

A Nontrivial Commutator

$$\bar{\mu}^*(x_{11}) = x_8 \otimes x_3$$

$$(\overline{\mu}^* - T^*\overline{\mu}^*)(x_{11}) = x_8 \otimes x_3 - x_3 \otimes x_8,$$

$$com^*(x_{11}) = x_8 \otimes x_3 - x_3 \otimes x_8$$

Complicated Commutators

$$H^*(E_8, \mu; \mathbb{F}_3) \cong \wedge(x_3, x_7, x_{15}, x_{19}, x_{27}, x_{35}, x_{39}, x_{47})$$
$$\otimes \mathbb{F}_3[x_8, x_{20}]/(x_8^3, x_{20}^3)$$

$$\bar{\mu}^*(x_{35}) = x_8 \otimes x_{27} - x_8^2 \otimes x_{19} + x_{20} \otimes x_{15} + x_8 x_{20} \otimes x_7$$

Complicated Commutators

$$(\overline{\mu}^* - T^*\overline{\mu}^*)(x_{35}) =$$

$$x_8 \otimes x_{27} - x_8^2 \otimes x_{19} + x_{20} \otimes x_{15} + x_8 x_{20} \otimes x_7$$

$$-x_{27} \otimes x_8 + x_{19} \otimes x_8^2 - x_{15} \otimes x_{20} - x_7 \otimes x_8 x_{20}$$

$$[\overline{x_8}, \overline{x_{27}}], \quad [\overline{x_8}^2, \overline{x_{19}}], \quad [\overline{x_{20}}, \overline{x_{15}}], \quad [\overline{x_8 x_{20}}, \overline{x_7}]$$

are nonzero in $H_*(E_8, \mu; \mathbb{F}_3)$

Complicated Commutators

$$\bar{\mu}^*(x_{35}) = x_8 \otimes x_{27} - x_8^2 \otimes x_{19} + x_{20} \otimes x_{15} + x_8 x_{20} \otimes x_7$$

$$(\overline{\mu}^* - T^*\overline{\mu}^*)(x_{35}) = com^*(x_{35})$$
$$= x_8 \otimes x_{27} - x_8^2 \otimes x_{19} + x_{20} \otimes x_{15} + x_8 x_{20} \otimes x_7$$
$$-x_{27} \otimes x_8 + x_{19} \otimes x_8^2 - x_{15} \otimes x_{20} - x_7 \otimes x_8 x_{20}$$

Observation

For these choices of generators,

$$\overline{\mu}^* - T^*\overline{\mu}^* = com^*$$

Generating Set for Cohomology

Theorem

(J. P. Lin) Let (X,μ) be a finite simply-connected HA-space and p be an odd prime. We can choose a generating set for $H^*(X,\mu;\mathbb{F}_p)$ so that if x is an element of this set, then

- *If x has even degree, $\overline{\mu}^*(x) = 0$.*
- *If x has odd degree, $\overline{\mu}^*(x) = \sum b \otimes r$ where each b is a product of even degree generators.*

Induced Homomorphism of com

Lemma

Let (X, μ) be a finite simply-connected HA-space and p be an odd prime. We can choose a generating set for $H^*(X, \mu; \mathbb{F}_p)$ so that if x is an element of this set, then

$$com^*(x) = \bar{\mu}^*(x) - T^*\bar{\mu}^*(x).$$

Hence $H_*(X, \mu; \mathbb{F}_p)$ is a commutative algebra iff com^* is a trivial homomorphism.

Observation

- $com^* + \mu^* = -\mu^* - T^*\mu^* = \frac{1}{2}(\mu^* + T^*\mu^*)$ on these choices of generators

Commutative Homology

Theorem

Let (X,μ) be a finite simply-connected HA-space and p be a fixed odd prime. Let

$$\nu(x,y) = \underbrace{((com(x,y)\ldots)\, com(x,y))}_{\frac{p-1}{2}\ times}\mu(x,y)$$

Then (X,ν) is an H-space for which $H_*(X,\nu;\mathbb{F}_p)$ is a commutative (non-associative) algebra.

Commutative Homology

Theorem

Furthermore, we can choose a generating set for $H^*(X, \nu; \mathbb{F}_p)$ so that if x is an element of this set,

$$\nu^*(x) = \frac{1}{2}\left(\mu^*(x) + T^*\mu^*(x)\right).$$

Examples

$$\nu(x,y) = xyx^{-1}y^{-1}xy$$

$$H^*(F_4, \nu; \mathbb{F}_3) \cong \wedge(x_3, x_7, x_{11}, x_{15}) \otimes \mathbb{F}_3[x_8]/(x_8^3)$$

$$\bar{\mu}^*(x_{11}) = x_8 \otimes x_3$$

$$\bar{\nu}^*(x_{11}) = -x_8 \otimes x_3 - x_3 \otimes x_8$$

Examples

$$H^*(E_8, \nu; \mathbb{F}_3) \cong \wedge(x_3, x_7, x_{15}, x_{19}, x_{27}, x_{35}, x_{39}, x_{47})$$
$$\otimes \mathbb{F}_3[x_8, x_{20}]/(x_8^3, x_{20}^3)$$

$$\bar{\mu}^*(x_{35}) = x_8 \otimes x_{27} - x_8^2 \otimes x_{19} + x_{20} \otimes x_{15} + x_8 x_{20} \otimes x_7$$

$$\bar{\nu}^*(x_{35}) = -x_8 \otimes x_{27} + x_8^2 \otimes x_{19} - x_{20} \otimes x_{15} - x_8 x_{20} \otimes x_7$$
$$-x_{27} \otimes x_8 + x_{19} \otimes x_8^2 - x_{15} \otimes x_{20} - x_7 \otimes x_8 x_{20}$$

Future Work

- When is $H_*(X, \nu; \mathbb{F}_p)$ commutative and associative?
- Is there a multiplication η on E_7 such that $H_*(E_7, \eta; \mathbb{F}_3)$ is commutative and associative?
- Is there a multiplication η on G such that $H_*(\Lambda G, \eta; \mathbb{F}_p)$ is commutative and associative?

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