Topological Triangulated Orbit Categories

Marcy Robertson
University of Western Ontraio

joint work with Andrew Salch and Julie Bergner

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Constructing Examples of Topological Cluster Categories

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Theorem (Happel '86)

D_Q admits a Serre functor, i.e. an autoequivalence $S : D_Q \to D_Q$ such that $DHom(X, ?) \cong Hom(?, SX)$ for all $X \in D_Q$, where $D = Hom_k(?, k)$.

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- (Kontsevich) T is d-Calabi-Yau if it has a Serre functor S and $S \cong \Sigma^d$ as triangle functors
- The **cluster category** C_Q is the universal 2-Calabi-Yau category under the derived category D_Q:

$$\begin{tikzcd} D_Q \arrow[d, "{(P,\pi)}"'] \arrow[dr, "{(F,\theta)}"] & \\ C_Q \arrow[r] & T \end{tikzcd}$$

- C_Q, T are 2-Calabi-Yau; P, F triangle functors; $\pi : P \circ S \to S \circ P$, $\theta : F \circ S \to S \circ F$

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$$C_Q(X, Y) = \bigoplus_{p \in Z} D_Q(X, (S^1 \circ \Sigma^2)^p Y).$$

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- Now, suppose T is a triangulated category and $F : T \to T$ is an autoequivalence. It still makes sense to construct an orbit category T/F with the same objects as T, but it is not usually the case that T/F is still a triangulated category or that the projection $P : T \to T/F$ is a triangulated functor.

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Example

For an example of when T/F is **not** triangulated, consider the following due to Neeman. Let $A = k[x]/(x^2)$ then $D^b(A)/\Sigma^2$ is not triangulated. To see this, note the following argument. Consider $1 + u \in End(k) \cong k[u]$. Then $1 + u$ is a monomorphism with no left inverse, but in a triangulated category all monomorphisms admit a left inverse.

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- So we will consider T as the homotopy category of k-module spectra.

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- Let X be an A-A-bimodule (**plus 'finiteness'**) and let $f = - \wedge_A X : mod - A \to mod - A$ such that $F = L(f) = - \wedge_A^L X : D(A) \to D(A)$ is an equivalence.

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- From here we can write down a description of $D(A)/F$, but we don't know if it is a triangulated category.

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- $B = A \oplus X[-1]$ where multiplication is given by the trivial extension.
- One shows that $D(B)/per(B)$ is the triangulated hull of the orbit category $D(A)/F$.

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- For the A-A-bimodule X we are assuming that X is 2-Calabi-Yau as a bimodule.
- This means that X satisfies duality properties, as a bimodule, which are similar to Dwyer-Miller duality.