The homotopy theory of coalgebras over a comonad

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Motivation

- Homotopic Hopf-Galois Extensions (Hess '09)
- Homotopic descent and codescent (Hess '10)

Want to form injective resolutions (fibrant replacements) in the category of coalgebras.

Definitions

Let (K, Δ, ε) be a comonad on $\mathbf{C}$ with $K : \mathbf{D} \to \mathbf{D}$, $\Delta : K \to K \circ K$, and $\varepsilon : K \to Id_{\mathbf{C}}$.

$\mathbf{C}_K$ is the category of K-coalgebras in $\mathbf{C}$ with objects (X, δ) with co-action $\delta : X \to KX$.

Let U_K be the forgetful functor

$$\mathbf{C}_K \underset{F_K}{\overset{U_K}{\rightleftarrows}} \mathbf{C}$$

with right adjoint F_K the cofree K-coalgebra with $F_K(X) = KX$, and $KX \xrightarrow{\Delta} KKX$.

Goal

Construct a model structure on the category of K-coalgebras, $\mathbf{M}_K$, with weak equivalences and cofibrations determined in $\mathbf{M}$ by the forgetful functor, U_K.

$$\mathbf{M}_K \underset{F_K}{\overset{U_K}{\rightleftarrows}} \mathbf{M}$$

Call such a structure a *left-induced* model structure.

Many examples, almost all involve chain complexes over a field.

Could use ∞-categories, wouldn't give explicit resolutions.

Postnikov Presentations

By adjunction, if a left-induced model structure exists, then K must take trivial fibrations to weak equivalences. Furthermore, pullbacks and inverse limits of such maps must also be weak equivalences.

Hess ('09) developed a theory of *Postnikov presentations* in model categories and used this to list general conditions for when such a model structure exists on $\mathbf{M}_K$.

The main condition is that any map constructed via pullbacks and inverse limits from K applied to trivial fibrations is a weak equivalence.

Theorem

(Hess - S.) Let $\mathbf{M}$ *be a model category with a Postnikov presentation. If* K *and* $\mathbf{M}$ *satisfy seven axioms then a left-induced model structure exists on* $\mathbf{M}_K$*.*

(K0) $\mathbf{M}_{\mathbb{K}}$ is complete.

(K1) $\delta : D \to KD$ is a cofibration for all $\mathbb{K}$-coalgebras (D, δ).

(K2) K preserves cofibrations.

(K3) Condition on products and cofibrations.

(K4) Condition on pullbacks and weak equivalences.

(K5) Condition on inverse limits and weak equivalences.

(K6) Condition on pullbacks, pushouts and weak equivalences.

Monomorphsims

Proposition (Adámek)

Let (K, Δ, ε) be a comonad on a well-powered category $\mathbf{C}$. If K preserves monomorphisms, then $\mathbf{C}_K$ is complete.

Proposition

Let $\mathbf{M}$ be a well-powered model category with cofibrations the monomorphisms. If K preserves monomorphisms, then axioms (K0) through (K2) hold.

Limits

Recall the later axioms all have to do with limits.

(K3) Condition on products and cofibrations.

(K4) Condition on pullbacks and weak equivalences.

(K5) Condition on inverse limits and weak equivalences.

(K6) Condition on pullbacks, pushouts and weak equivalences.

Easiest when limits in $\mathbf{M}_K$ are created in $\mathbf{M}$.

D.G. setting

Let R be a ring and A be a differential graded R-algebra.

Let $\mathbf{M}_A$ be the category of right A-modules with cofibrations the monomorphisms and weak equivalences the quasi-isomorphisms.

Let V be an A co-ring (comonoid in A-bimodules).

Let M_A^V be the category of right V-comodules in $\mathbf{M}_A$.

$$\mathbf{M}_A^V \underset{-\otimes_A V}{\overset{U}{\rightleftarrows}} \mathbf{M}_A$$

Semifree V

V is *semi-free* if $V_\natural \cong A_\natural \otimes X$ as an underlying A-module (without the differential). Here X is a graded R-module.

Proposition

Assume V is an A-co-ring that is semifree as a left A-module. Then $- \otimes_A V$ preserves monomorphisms.

Proposition

Assume V is an A-co-ring that is semifree as a left A-module on a generating graded R-module X such that X_n is R-free and finitely generated for all $n \geq 0$. Then all limits in $\mathbf{M}_A^V$ are created in $\mathbf{M}_A$.

Limits

Proposition

Assume V is an A-co-ring that is semifree as a left A-module on a generating graded R-module X such that X_n is R-free and finitely generated for all $n \geq 0$. Then all limits in $\mathbf{M}_A^V$ are created in $\mathbf{M}_A$.

Sketch of Proof:

Due to the semifree property, $- \otimes_A V$ preserves kernels; hence also pullbacks.

Due to the hypotheses on X, $- \otimes_A V$ commutes with arbitrary products.

(K4) Condition on pullbacks and weak equivalences

For every pullback diagram in $\mathbf{M}$

$$\begin{tikzcd} E \times_B D \arrow[r] \arrow[d, two heads] & E \arrow[d, two heads, "p"', "\sim_n"] \\ D = U_{\mathbb{K}}(D, \delta) \arrow[r, "f"] & B \end{tikzcd}$$

the induced morphism $U_{\mathbb{K}}\big(F_{\mathbb{K}}E \times_{F_{\mathbb{K}}B} (D, \delta)\big) \to E \times_B D$ is an $n+1$-equivalence.

For a "toy case," set $B = 0 = D$. Show if $F \xrightarrow{\sim_n} 0$ then $F \otimes_A V \to F$ is an $n+1$ equivalence.

(K4) Condition on pullbacks and weak equivalences.

For a "toy case," set $B = 0 = D$. Show if $F \xrightarrow{\sim_n} 0$ then $F \otimes_A V \to F$ is an $n + 1$ equivalence.

Use flatness and connectivity conditions on $R \otimes_A V$ to ensure this result.

Use the toy case to prove the general pullback condition.

Theorem (Hess - S.)

Let R be a semihereditary commutative ring and A an augmented dg R-algebra such that $H_1 A = 0$. If V is an A-co-ring that is semifree as a left A-module on a generating graded R-module X such that

1. $H_0(R \otimes_A V) = R$, $H_1(R \otimes_A V) = 0$, *and*
2. X_n *is R-free and finitely generated for all* $n \geq 0$,

then the category $\mathcal{M}_A^V$ admits a model category structure left-induced from $\mathcal{M}_A$ by the adjunction

$$\mathcal{M}_A^V \underset{-\otimes_A V}{\overset{U}{\rightleftarrows}} \mathcal{M}_A.$$

Remarks on (K5) and (K6)

(K5) Condition on inverse limits and weak equivalences.

Basically follows by Mittag-Leffler condition.

(K6) Condition on pullbacks, pushouts and weak equivalences.

Basically follows from (K4) and stability.

References

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