The Third and the Automatic Homotopy Exact Sequence of a Fibration in Module Theory

In Memory of Peter Hilton

C. Joanna Su

ABSTRACT

The homotopy theory of modules was developed by Peter Hilton in the 1950s, as a natural analog to the existing homotopy theory in algebraic topology. One of the reasons that the study of the subject grew out of fashion in the late 1960s was that, while the concept of a fibration in module theory seemed intuitive and did induce a homotopy exact sequence, the ‘expected’ sequence - one that is parallel to and carries the same character as the homotopy exact sequence of a fibration in topology - failed to be discovered.

Our search shows that in module theory a fibration induces not just one, but three homotopy exact sequences - the first, the expected, and the automatic homotopy exact sequences of a fibration in module theory, respectively. Each sequence carries different features. The first and original sequence displays an isomorphism between the relative homotopy groups and the homotopy groups of the ‘fiber’. The expected sequence both displays an analogous appearance to the homotopy exact sequence of a fibration in topology and inherits the characteristic isomorphism between the relative homotopy groups and the homotopy groups of the induced ‘base space’.

In this talk, we discuss the third of the three homotopy sequences - the automatic homotopy exact sequence of a fibration - which takes place in the projective homotopy theory of modules, whereas the other two sequences took place in the injective homotopy theory of modules. It turns out that the automatic sequence displays an isomorphism between the relative homotopy groups and the ‘strong’ homotopy groups of the fiber.

INTRODUCTION

The homotopy theory of modules was developed by Peter Hilton in the 1950s as a result of his trip to Warsaw, Poland, to work with Karol Borsuk, and to Zurich, Switzerland, to work with Beno Eckmann. It produced a natural analog to the existing homotopy theory in algebraic topology. Compared and parallel to topological spaces and continuous maps in topology, in module theory the categorical objects are Λ-modules, where Λ is a unitary ring, and the morphisms are Λ-module homomorphisms (Λ-maps).

Λ: a unitary ring
A, B: right Λ-modules
φ: A $\to$ B: a Λ-module homomorphism

• **nullhomotopy**: φ is *i-nullhomotopic* if φ can be extended to an injective module, CA, containing A.

• **homotopy**: two maps are *i-homotopic* if their difference is *i*-nullhomotopic.

• **suspension**: A $\to$ CA $\to$ CA/A = ΣA (the *suspension* of A.)

• **homotopy group**: the *(injective) homotopy group* of maps of A to B is $\bar{\pi}(A, B) = \mathrm{Hom}_\Lambda(A, B)/\mathrm{Hom}_0(A, B)$, where $\mathrm{Hom}_0(A, B)$ is the set of *i*-nullhomotopic maps.

• **n^{th} homotopy group**: the *n^{th} i-homotopy group* of A to B is $\bar{\pi}_n(A, B) = \bar{\pi}(\Sigma^n A, B)$.

REMARK

Unlike the homotopy theory in topology, there are two homotopy theories in module theory - the injective homotopy theory and the projective homotopy theory. The two theories are dual to each other, but not isomorphic.

Back in the 1950s and 1960s when the study of the subject was much in fashion, after one discovered a theorem in one theory, one knew that there would be a dual result in the other theory. However, one would need to produce a separate proof for the dual theorem. Even though these dual proofs seemed duplicative to the originals, they were necessary to ensure that the dual results indeed held true.

During our study and search within this subject of homotopy theory of modules, we developed a way of delivering proofs without use of any elements of sets in the arguments.

Precisely, instead of tracing the elements through maps and commutative diagrams, we exploit only the maps themselves and trace them through the commutative diagrams in question.

Since our arguments do not involve any elements of sets, by duality we can proceed and state the dual theorems in the other theory without further derivations.

Not only does this method avoid the need of those duplicative, separate proofs for the dual theorems, but also these original proofs seem surprisingly intuitive and simple, once they are found.

FIBRATION

In Topology

- E, B, X: topological spaces
- I: unit interval
- A surjection p:E→B is a *fibration* if it has the homotopy lifting property with respect to every X.

In Module Theory

- E, B, X: right Λ-modules
- I: nonzero injective right Λ-module
- An epimorphism p:E→B is a *fibration* if it has the homotopy lifting property with respect to every X.

In Topology

• A map p:E→B is said to have the *homotopy lifting property* with respect to X if

$$\begin{tikzcd} & X \arrow[dl, "g"'] \arrow[d, "p\circ g"] \\ E \arrow[r, "p"'] & B \end{tikzcd} \qquad \begin{tikzcd} & X \times I \arrow[dl, dashed, "F"'] \arrow[d, "G"] \arrow[dl, phantom, shift right=3ex, "\circlearrowleft"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

In Module Theory

• A map p:E→B is said to have the *homotopy lifting property* with respect to X if

$$\begin{tikzcd} & X \arrow[dl, "g"'] \arrow[d, "p\circ g"] \\ E \arrow[r, "p"'] & B \end{tikzcd} \qquad \begin{tikzcd} & X \times I \arrow[dl, dashed, "F"'] \arrow[d, "G"] \arrow[dl, phantom, shift right=3ex, "\circlearrowleft"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

$$\begin{tikzcd} & I \arrow[dl, dashed, "\delta"'] \arrow[d, "\phi"] \arrow[dl, phantom, shift right=3ex, "\circlearrowleft"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

Definition: Let Λ be a unitary ring and E, B be right Λ-modules. We say that a Λ-module homomorphism (Λ-map) $p : E \to B$ has the *homotopy lifting property* with respect to I, a non-zero injective right Λ-module, if for every Λ-map $\phi : I \to B$ there exists a Λ-map $\delta : I \to E$ such that $\phi = p \circ \delta$.

$$
\begin{tikzcd}
 & I \arrow[dl, dashed, "\delta"'] \arrow[d, "\phi"] \\
E \arrow[r, "p"'] & B \arrow[ul, phantom, "\circlearrowleft", pos=0.35]
\end{tikzcd}
$$

An epimorphism $p : E \twoheadrightarrow B$ is said to be a *fibration* if it has the homotopy lifting property with respect to every injective I.

HOMOTOPY EXACT SEQUENCE OF A FIBRATION

In Topology

p: E → B is a fibration

E, B: topological spaces

$b_0 \in B$, $e_0 \in F$

F is the fiber over b_0

$$\cdots \longrightarrow \pi_n(F, e_0) \longrightarrow \pi_n(E, e_0) \longrightarrow \pi_n(B, b_0)$$

$$\longrightarrow \pi_{n-1}(F, e_0) \longrightarrow \pi_{n-1}(E, e_0) \longrightarrow \cdots$$

In Module Theory

p: E → B is a fibration

E, B: right Λ-modules

A: arbitrary right Λ-module

F is the kernel/fiber of p

$$\cdots \longrightarrow \overline{\pi}_n(A, F) \longrightarrow \overline{\pi}_n(A, E) \xrightarrow{p_*} \overline{\pi}_n(A, B)$$

$$\longrightarrow \overline{\pi}_{n-1}(A, F) \longrightarrow \overline{\pi}_{n-1}(A, E) \xrightarrow{p_*} \cdots$$

In Topology

The relative homotopy group $\pi_n(E, F, e_0)$ ***IS*** isomorphic to the homotopy group of the base space $\pi_n(B, b_0)$, n $\geq$1.

In Module Theory

The relative homotopy group $\bar{\pi}_n(A, p)$ ***IS NOT*** isomorphic to the homotopy group of the 'base' $\overline{\pi}_n(A, B)$ n $\geq$1.

A SUBTLE & CRUCIAL TWIST

• Given a Λ-map $p: E \rightarrow B$.

• Since a fibration needs to be an epimorphism, we apply the mapping cylinder of p in order to produce an epimorphism κ in its resulting short exact sequence

$$E \rightarrowtail^{\{\iota, p\}} CE \oplus B \xrightarrow{\kappa} \twoheadrightarrow D.$$

(CE is an injective container of E with the inclusion map $\iota: E \rightarrow CE$, κ is the quotient map, and $D = \mathrm{coker}\{\iota, p\}$.)

• We then say that the map p *induces* a fibration if its induced epimorphism κ is a fibration.

• As we shift the emphasis from p to κ, that is, if p induces a fibration, there arises ***the expected homotopy exact sequence of a fibration in module theory***:

$$\cdots \longrightarrow \overline{\pi}_n(A,E) \xrightarrow{p_*} \overline{\pi}_n(A,B) \xrightarrow{\kappa_*} \overline{\pi}_n(A,D) \longrightarrow \overline{\pi}_{n-1}(A,E) \xrightarrow{p_*} \cdots .$$

This is the long-sought-for sequence.

The Expected Homotopy Exact Sequence of a Fibration in Module Theory

$$\cdots \longrightarrow \overline{\pi}_n(A,E) \xrightarrow{p_*} \overline{\pi}_n(A,B) \xrightarrow{\kappa_*} \overline{\pi}_n(A,D) \longrightarrow \overline{\pi}_{n-1}(A,E) \xrightarrow{p_*} \cdots$$

- It displays an analogous appearance to the homotopy exact sequence of a fibration in topology.

- It demonstrates the desired characteristic isomorphism between the relative homotopy group $\overline{\pi}_n(A,p)$ and the homotopy group of the newly induced 'base space' $\overline{\pi}_n(A,D)$, $n \geq 1$.

Our study shows that in module theory a fibration induces not just one, but three homotopy exact sequences.

Based on the character of each individual, they are named as the first, the expected, and the automatic homotopy exact sequences of a fibration in module theory, respectively.

HOMOTOPY EXACT SEQUENCE OF A FIBRATION

In Topology

p: E → B is a fibration

E, B: topological spaces

$b_0 \in B$, $e_0 \in F$

F is the fiber over b_0

$$\cdots \longrightarrow \pi_n(F, e_0) \longrightarrow \pi_n(E, e_0) \longrightarrow \pi_n(B, b_0)$$

$$\longrightarrow \pi_{n-1}(F, e_0) \longrightarrow \pi_{n-1}(E, e_0) \longrightarrow \cdots$$

In Module Theory

p: E → B is a fibration

E, B: right Λ-modules

A: arbitrary right Λ-module

F is the kernel/fiber of p

$$\cdots \longrightarrow \overline{\pi}_n(A, F) \longrightarrow \overline{\pi}_n(A, E) \xrightarrow{p_*} \overline{\pi}_n(A, B)$$

$$\longrightarrow \overline{\pi}_{n-1}(A, F) \longrightarrow \overline{\pi}_{n-1}(A, E) \xrightarrow{p_*} \cdots$$

The First and the Original Homotopy Exact Sequence of a Fibration in Module Theory

$$\cdots \longrightarrow \overline{\pi}_n(A,F) \longrightarrow \overline{\pi}_n(A,E) \xrightarrow{p_*} \overline{\pi}_n(A,B) \longrightarrow \overline{\pi}_{n-1}(A,F) \longrightarrow \cdots$$

- This sequence features an interesting phenomenon that the relative homotopy group $\overline{\pi}_n(A,p)$ is isomorphic to the homotopy group of the 'fiber' $\overline{\pi}_{n-1}(A,F)$, n ≥ 1, where F is the kernel/fiber of the fibration $p : E \twoheadrightarrow B$.

These two sequences take place in the injective homotopy theory of modules.

The third of the three homotopy sequences - ***the automatic homotopy exact sequence of a fibration in module theory*** - takes place in the projective homotopy theory of modules.

FIBRATION

In the Injective Theory

- E, B, X: right Λ-modules
- I: nonzero injective right Λ-module
- An epimorphism p:E→B is a *fibratio*n if it has the homotopy lifting property with respect to every X.

In the Projective Theory

- E, B, X: left Λ-modules
- P: nonzero projective left Λ-module
- An epimorphism p:E→B is a *fibration* if it has the homotopy lifting property with respect to every X.

In the Injective Theory

- A map p:E→B is said to have the *homotopy lifting property* with respect to X if

$$\begin{tikzcd} & X \arrow[dl, "g"'] \arrow[d, "p\circ g"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

$$\begin{tikzcd} & X \times I \arrow[dl, dashed, "F"'] \arrow[d, "G"] \arrow[dl, phantom, shift right=4ex, "\circlearrowleft"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

$$\begin{tikzcd} & I \arrow[dl, dashed, "\delta"'] \arrow[d, "\phi"] \arrow[dl, phantom, shift right=4ex, "\circlearrowleft"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

In the Projective Theory

- A map p:E→B is said to have the *homotopy lifting property* with respect to X if

$$\begin{tikzcd} & X \arrow[dl, "g"'] \arrow[d, "p\circ g"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

$$\begin{tikzcd} & X \arrow[dl, dashed, "\exists H"'] \arrow[d, "G"] \arrow[dl, phantom, shift right=4ex, "\circlearrowleft"] \\ E \times P \arrow[r, "\tilde{p}"'] & B \times P \end{tikzcd}$$

Proposition: Let Λ be a unitary ring and E, B, X be left Λ-modules.

Every Λ-map $p : E \to B$ holds the homotopy lifting property with respect to every X.

Thus, every epimorphism $p : E \twoheadrightarrow B$ is ***automatically*** a fibration.

- Given an epimorphism $p : E \twoheadrightarrow B$.
- It is automatically a fibration.
- Expand $p : E \twoheadrightarrow B$ to the short exact sequence

$$F \overset{\iota}{\hookrightarrow} E \overset{p}{\twoheadrightarrow} B,$$

where F is the fiber/kernel of the fibration p.

The Automatic Homotopy Exact Sequence of a Fibration in Module Theory

$$\cdots \longrightarrow \underline{\pi}_n(B,A) \xrightarrow{p^+} \underline{\pi}_n(E,A) \longrightarrow S\underline{\pi}_n(F,A) \longrightarrow \underline{\pi}_{n-1}(B,A) \xrightarrow{p^+} \cdots$$

- This sequence features an isomorphism between the relative homotopy group $\underline{\pi}_n(p,A)$ and the ‘strong’ homotopy group $S\underline{\pi}_n(F,A)$ of the fiber F, $n \geq 1$.

The ‘Strong’ Homotopy Group $S\underline{\pi}_n(F, A)$ of the Fiber

• It has very special double-features.

• It inherits all the properties of a relative homotopy group $\underline{\pi}_n(p, A)$.

• It is isomorphic to a subgroup of the homotopy group of the fiber $\underline{\pi}_n(F, A)$.

• It carries the characteristics of an absolute *p*-homotopy group.

FINAL NOTE

Since our arguments do not involve any elements of sets, by duality we can proceed and state the dual theorems in the other theory without further derivations.

Not only does this method avoid the need of those duplicative, separate proofs for the dual theorems, but also these original proofs seem surprisingly intuitive and simple, once they are found.

Now apply the duality!