Posets of Partitions, Row Shuffles and Schur positivity

A well–known integral basis for the ring of symmetric functions in n variables is the set of Schur functions s_λ, where λ varies over partitions with at most n parts. A symmetric function is said to be Schur positive if it can be written as non–negative integer linear combination of Schur functions. Fomin, Fulton, Li and Poon defined the notion of a row shuffle of a pair of partitions: given two partitions λ and μ, the row shuffle is a pair (ν_1, ν_2) partitions obtained by shuffling the rows of the Young diagrams of λ and μ. They conjectured (now a theorem due to T. Lam, A. Postnikov and P. Pylyavskyy, that the difference $s_{\nu_1} s_{\nu_2} - s_\lambda - s_\mu$ is Schur positive.

In terms of representation theory of the general linear algebra $\mathfrak{gl}_n$ this question amounts to asking when the following to holds:

$$\dim \operatorname{Hom}_{\mathfrak{g}}(V, V(\lambda) \otimes V(\mu)) \leq \dim \operatorname{Hom}_{\mathfrak{g}}(V, V(\nu_1) \otimes V(\nu_2)),$$

where V is an arbitrary irreducible representation of $\mathfrak{gl}_n$ and $V(\lambda)$ is the irreducible finite–dimensional $\mathfrak{g}$–module whose character is s_λ. In this talk we shall discuss a partial order on pairs (more generally k–tuples) of partitions which " add " up to the same partition. We shall see that the maximal element in this partial order coincides with the row shuffle of partitions defined by Fomin, Fulton, Li and Poon. In joint work with Fourier and Sagaki, we conjecture that that the inequality discussed above holds along the partial order. We shall discuss the cases when the conjecture is known to be true.