L_∞ and A_∞ Algebras

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Setting

A differential graded vector space, for homotopy.

Definition

Let A be a $\mathbb{Z}$-graded vector space $A = \bigoplus_{r\in\mathbb{Z}} A^r$ and suppose that there exists a collection of degree one multi-linear maps

$$m := \{m_k : A^{\otimes k} \to A\}_{k\geq 1}$$

(A, m) is called an A_∞ algebra when the multi-linear maps m_k satisfy the following relations

$$\sum_{k+l=n+1} \sum_{i=1}^{k} (-1)^{o_1+\cdots+o_{i-1}} m_k(o_1, \ldots, o_{i-1}, m_l(o_i, \ldots, o_{i+l-1}), o_{i+l}, \ldots, o_n) = 0 \tag{1}$$

for $n \geq 1$, where o_j on (-1) denotes the degree of o_j.

Definition

Definition (L_∞ Algebra)

Let L be a graded vector space and suppose that a collection of degree one graded symmetric linear maps $l := \{l_k : L^{\wedge k} \to L\}_{k \geq 1}$ is given. (L, l) is called an L_∞ algebra if and only if the maps satisfy the following relations:

$$\sum_{\sigma \in S_{k+l=n}} (-1)^{\epsilon(\sigma)} l_{1+l}(l_k(c_{\sigma(1)}, \ldots, c_{\sigma(k)}), c_{\sigma(k+1)}, \ldots, c_{\sigma(n)}) = 0 \qquad (2)$$

for $n \geq 1$, where $(-1)^{\epsilon(\sigma)}$ is the Kozsul sign of the permutation.

Maps

$n = 3:$

For $v_1 \otimes w^{\otimes 2}$:

$$(-1)^0(-1)^0 l_3(l_1(v_1), w_1, w_2) + (-1)^{0\cdot 1}(-1)^1 l_3(l_1(w_1), v_1, w_2) +$$
$$(-1)^{0\cdot 1}(-1)^{1\cdot 1}(-1)^2 l_3(l_1(w_2), v_1, w_1) + (-1)^0(-1)^0 l_2(l_2(v_1, w_1), w_2) +$$
$$(-1)^{1\cdot 1}(-1)^1 l_2(l_2(v_1, w_2), w_1) + (-1)^{0\cdot 1}(-1)^{0\cdot 1}(-1)^2 l_2(l_2(w_1, w_2), v_1) +$$
$$(-1)^0(-1)^0 l_1(l_3(v_1, w_1, w_2)) = \dots$$

A_∞

Define the maps $m_n : V^{\otimes n} \to V$ by

$$m_1(v_1) = m_1(v_2) = w$$

For $n \geq 2 : m_n(v_1 \otimes w^{\otimes k} \otimes v_1 \otimes w^{\otimes (n-2)-k}) = (-1)^k s_n v_1,\ 0 \leq k \leq n-2$

$$m_n(v_1 \otimes w^{\otimes (n-2)} \otimes v_2) = s_{n+1} v_1$$

$$m_n(v_1 \otimes w^{\otimes (n-1)}) = s_{n+1} w$$

where $s_n = (-1)^{\frac{(n+1)(n+2)}{2}}$, and $m_n = 0$ when evaluated on any element of $V^{\otimes n}$ that is not listed above.

A Finite L_∞ Example

Consider the graded vector space $V = V_0 \oplus V_1$ where V_0 has basis $< v_1, v_2 >$ and V_1 has basis $< w >$. We show that this space has an L_∞ structure given by:

$$\begin{aligned} l_1(v_1) = l_1(v_2) &= w \\ l_2(v_1 \otimes v_2) &= s_3 v_1 \\ l_2(v_1 \otimes w) &= s_3 w \\ \text{For } n \geq 3 \;\; l_n(v_1 \otimes w^{\otimes(n-1)}) &= (n-1)! s_{n+1} w \\ l_n(v_1 \otimes w^{\otimes(n-2)} \otimes v_2) &= (n-2)! s_{n+1} v_1 \end{aligned}$$

where $s_n = (-1)^{\frac{(n+1)(n+2)}{2}}$ and $l_n = 0$ when evaluated on any element of $V^{\otimes n}$ that is not listed.

Strong Homotopy Derivations

Definition

A *strong homotopy derivation* of degree one of an A_∞-algebra (A, m) consists of a collection of multi-linear maps of degree one

$$\theta := \{\theta_q | A^{\otimes q} \to A\}_{q \geq 1}$$

satisfying the following relations:

$$0 = \sum_{r+s=q+1} \sum_{i=0}^{r-1} (-1)^{\beta(s,i)} \theta_r(o_1, \dots, o_i, m_s(o_{i+1}, \dots, o_{i+s}), \dots, o_q) + (-1)^{\beta(s,i)} m_r(o_1, \dots, o_i, \theta_s(o_{i+1}, \dots, o_{i+s}), \dots, o_q) \quad (3)$$

Here the sign $\beta(s, i) = o_1 + \cdots + o_i$ results from moving m_s, respectively θ_s, past $(o_1, \dots, o_i)$.

Strong Homotopy Derivations

From a comment, we show $[m, \theta] = 0$ is equivalent to (3).

$$[m, \theta] = m \circ \theta - (-1)^{|m||\theta|}\theta \circ m = m \circ \theta + \theta \circ m$$

Strong Homotopy Derivations for L_∞ Algebras

Definition

A *strong homotopy derivation* of degree one of an L_∞ algebra consists of a collection of multi-linear maps of degree one

$$\theta := \{\theta_q | L^{\wedge q} \to L\}_{q \geq 1}$$

satisfying relations:

$$\sum_{\substack{j=1 \\ \sigma \in U(j,n-j)}}^{j=n} (-1)^{\epsilon(\sigma)} \theta_{n-j+1}(l_j(x_{\sigma(1)}, \ldots, x_{\sigma(j)}), x_{\sigma(j+1)}, \ldots, x_{\sigma(n)}) + (-1)^{\epsilon(\sigma)} l_{n-j+1}(\theta_j(x_{\sigma(1)}, \ldots, x_{\sigma(j)}), x_{\sigma(j+1)}, \ldots, x_{\sigma(n)}) \quad (4)$$

where $(-1)^{\epsilon(\sigma)}$ is the product of the permuted elements.

Derivations

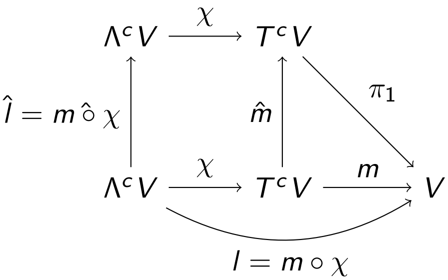
$$[\theta, l] = \theta \circ l - (-1)^{|\theta||l|} l \circ \theta = \theta \circ l + l \circ \theta$$

$$x \wedge y = (-1)^{|x||y|} y \wedge x$$

$$\begin{aligned}
[\theta, l](x, y) &= \theta(l_2(x, y) + l_1(x) \wedge y + (-1)^{|x||y|} l_1(y) \wedge x) \\
&\quad + l(\theta_2(x, y) + \theta_1(x) \wedge y + (-1)^{|x||y|} \theta_1(y) \wedge x) \\
&= \theta_1 l_2(x, y) + \theta_2(l_1(x), y) + \theta_1 l_1(x) \wedge y \\
&\quad + (-1)^{|l_1(x)||y|} \theta_1(y) \wedge l_1(x) + (-1)^{|x||y|} \theta_2(l_1(y), x) \\
&\quad + (-1)^{|x||y|} \theta_1 l_1(y) \wedge x + (-1)^{|x||y| + |l_1(y)||x|} \theta_1(x) \wedge l_1(y) \\
&\quad + l_1 \theta_2(x, y) + l_2(\theta_1(x), y) + l_1 \theta_1(x) \wedge y + \\
&\quad (-1)^{|\theta_1(x)||y|} l_1(y) \wedge \theta_1(x) + (-1)^{|x||y|} l_2(\theta_1(y), x) \\
&\quad + (-1)^{|x||y|} l_1 \theta_1(y) \wedge x + (-1)^{|x||y| + |\theta_1(y)||x|} l_1(x) \wedge \theta_1(y)
\end{aligned}$$

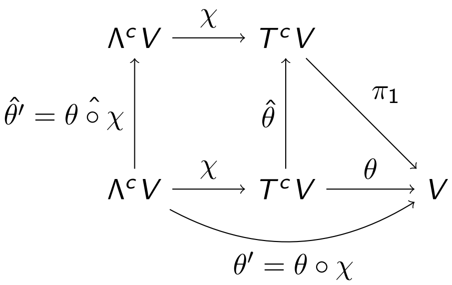
Relating Strong Homotopies

Let (V, m) be an A_∞ structure and extend this to an L_∞ structure given by (V, l), where l is found by skew-symmetrizing m, as we did before in Theorem 3. This gives us the diagram:



Relating Strong Homotopies

Now we let (V, θ) give a strong homotopy derivation structure and we define θ' to be the skew-symmetriczation of θ, again using Theorem 3. This gives the picture:



Derivations

To show that $[\hat{l}, \hat{\theta}'] = 0$:

$$\begin{aligned}
\chi[\hat{l}, \hat{\theta}'] &= \chi(\hat{l}\hat{\theta}' + \hat{\theta}'\hat{l}) \\
&= \chi\hat{l}\hat{\theta}' + \chi\hat{\theta}'\hat{l} \\
&= \hat{m}\chi\hat{\theta}' + \hat{\theta}\chi\hat{l} \\
&= \hat{m}\hat{\theta}\chi + \hat{\theta}\hat{m}\chi \\
&= [\hat{m}, \hat{\theta}]\chi \\
&= 0
\end{aligned}$$

Example of A_∞ Algebra HD

Jacobson [6] defines an *inner derivation* to fix an element a, then

$$D_a(x) = xa - ax$$

Define θ_n for a generic n and show this works with (3). Define

$$\theta_n(x_1, \ldots, x_n) = m_{n+1}(x_1, \ldots, x_n, a) + m_{n+1}(x_1, \ldots, x_{n-1}, a, x_n)$$

$$+ \cdots + m_{n+1}(x, a, x_2, \ldots, x_n) + m_{n+1}(a, x_1, \ldots, x_n)$$

where $m_1(a) = 0$ and $|a| = 2k$ for some $k \in \mathbb{Z}$.

L_∞ Algebra HD

Let $\theta_n(x_1, \ldots, x_n) := l_{n+1}(x_1, \ldots, x_n, a)$ where $l_1(a) = 0$ and $|a| = 2k$ for some $k \in \mathbb{Z}$.

Concrete A_∞ Homotopy Derivation Example

Example

Let $V = V_{-1} + V_0$ be given by $V_{-1} = \langle v_1, v_2 \rangle$ and $V_0 \langle w \rangle$, where an A_∞ algebra structure has been given by

$$\begin{aligned} \hat{m}_1(v_1) = \hat{m}_1(v_2) &= w \\ \text{For } n \geq 2 \ \hat{m}_n(v_1 \otimes w^{\otimes k} \otimes v_1 \otimes w^{n-2-k}) &= v_1 \text{ for } 0 \leq k \leq n-2 \\ \hat{m}_n(v_1 \otimes w^{\otimes n-2} \otimes v_2) &= v_1 \\ \hat{m}_n(v_1 \otimes w^{\otimes n-1}) &= w \end{aligned}$$

Then the following gives a strong homotopy derivation on this coalgebra:

$$\begin{aligned} \theta_1(v_1) &= w \\ \text{For } n \geq 2 \ \theta_n(v_1 \otimes w^{\otimes k} \otimes v_1 \otimes w^{n-2-k}) &= n v_1 \text{ where } 0 \leq k \leq n-2 \\ \theta(v_1 \otimes w^{\otimes n-1}) &= nw \end{aligned}$$

Concrete L_∞ Homotopy Derivation Example

Example

Let $W = W_{-1} + W_0$ be given by $W_{-1} = \langle x_1, x_2 \rangle$ and $W_0 = \langle y \rangle$ with maps given by $\hat{l}_n : W^{\wedge n} \to W$ where

$$\begin{aligned} \hat{l}_1(x_1) = \hat{l}_1(x_2) &= y \\ \hat{l}_n(x_1 \wedge y^{\wedge n-1} &= (-1)^{n^2+1}(n-1)!y \\ \hat{l}_n(x_1 \wedge y^{n-2} \wedge x_2) &= (-1)^{n^2+1}(n-2)!x_1 \end{aligned}$$

as an L_∞ structure. Then a strong homotopy derivation on W is given by the following symmetric maps $\hat{\theta} : W^{\wedge n} \to W$:

$$\begin{aligned} \hat{\theta}_1(x_1) &= y \\ \hat{\theta}_n(x_1 \wedge y^{\wedge n-1}) &= (-1)^{n^2} n!y \\ \hat{\theta}_n(x_1 \wedge y^{\wedge n-2} \wedge x_2) &= (-1)^{n^2}(n-1)!x_1 \end{aligned}$$

Current Work

- Continue studying derivations and the connection between typical algebras and those that have been suspended.

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