Azumaya Orders do not Always Exist

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Azumaya Algebras over Rings

Overview

- Azumaya algebras over schemes (and rings) generalize central simple algebras over fields.
- They do not satisfy the same existence theorems.

Examples of Central Simple Algebras over a field

- $\mathbb{H}$ is a central division algebra (CDA) over $\mathbb{R}$
- If D is a CDA over K, then $\mathrm{Mat}_n(D)$ is central simple over K
  Artin-Wedderburn theory says this is the only example.

Generalization to local rings by Azumaya (Azu51), and to all rings by Auslander & Goldman (AG60).

Azumaya Algebras over Schemes

Generalization to schemes by Grothendieck, (Gro68).

An Azumaya algebra $\mathcal{A}$ over a scheme X is

- A coherent locally-free sheaf of $\mathcal{O}_X$-algebras.
- Étale-locally isomorphic to a sheaf of matrix algebras.

Note that $\mathrm{rank}(\mathcal{A}) = n^2$ for some integer: n is called the degree of $\mathcal{A}$.

- When $X = \operatorname{Spec} R$ is an affine scheme, this recaptures the Auslander-Goldman definition.
- Over a field, it demands that A be a finite-dimensional k-algebra such that $A \otimes_k k^{\mathrm{sep}}$ is a matrix algebra.

Locally Matrix Algebras

There is a bijective correspondence

$$\{\text{degree-}n \text{ Azumaya algebras}\}/\cong \longleftrightarrow \{\text{ principal } \mathrm{PGL}_n\text{-bundles}\}/\cong$$

The following always exist:

- Matrix algebras over any field.
- Azumaya algebras $\mathrm{End}(\mathcal{V})$, where $\mathcal{V}$ is a vector-bundle on X.

Definition

The Brauer group of X is the groupification of the monoid of Azumaya algebras under $\otimes$, subject to the further relation that $\mathrm{End}(\mathcal{V}) \sim 0$.
It is denoted $\mathrm{Br}(X)$.

Definition

Two Azumaya algebras, $\mathcal{A}$ and $\mathcal{A}'$ are Brauer equivalent (we'll just say 'equivalent') if their classes $[\mathcal{A}]$ and $[\mathcal{A}']$ in $\mathrm{Br}(X)$ agree.

The Period and Index

We consider henceforth only regular schemes over a field.

Facts about Brauer classes

- The group $\mathrm{Br}(X)$ is a torsion group.
- The period of $[\mathcal{A}]$ is the order of $[\mathcal{A}]$ in $\mathrm{Br}(X)$.
- The index is the GCD of all the degrees of all the Azumaya algebras equivalent to $\mathcal{A}$.
- The period divides the index, and they have the same prime divisors.

Theorem

Over a field, $\mathrm{ind}([\mathrm{Mat}_n(D)])$ *is the degree of the CDA* $D \sim \mathrm{Mat}_n(D)$

Over a field, there is an algebra of minimal degree in each equivalence class.

Azumaya Orders

- X be a regular, noetherian, integral scheme.
- K the fraction-field of X.

Definition

If A is a central simple algebra over K, we say $\mathcal{A}$ is an Azumaya order for A if $\mathcal{A}$ is Azumaya and

$$\mathcal{A} \otimes_{\mathcal{O}_X} K \cong A$$

When do Azumaya orders exist?

- $[A]$ has to be in the image of the map $\mathrm{Br}(X) \longrightarrow \mathrm{Br}(K)$.
- **Auslander-Goldman, 1960:** This is the only restriction for $X = \operatorname{Spec} R$ of dimension ≤ 2.

Auslander and Goldman also prove the map $\mathrm{Br}(\operatorname{Spec} R) \longrightarrow \mathrm{Br}(K)$ is injective in general.

Non-existence

Theorem (Antieau-W., 2012; Main Theorem.)

There exists a regular integral domain R and a class $[\mathcal{A}]$ in $\mathrm{Br}(\mathrm{Spec}\, R)$ such that $\mathrm{ind}([\mathcal{A}]) = 2$, but such that there are no degree-2 Azumaya algebras over R equivalent to $\mathcal{A}$.

And a corollary:

Theorem (Antieau-W., 2012)

There exists a regular integral domain R and a class $[A]$ in the image of $f^* : \mathrm{Br}(\mathrm{Spec}\, R) \to \mathrm{Br}(K)$ for which A does not posses an Azumaya order.

Proof, given first theorem.

The map f^* is injective. The class $f^*([\mathcal{A}])$ has period, index 2.
There is a degree-2 division algebra in the class of $f^*([\mathcal{A}])$, which cannot extend to an Azumaya algbera on Spec R. $\square$

Analogous Definitions

Our main tool is comparison of algebra and topology.

Suppose X is a topological space

Topological analogues

- Azumaya algebras on X: bundles of rings locally isomorphic to $\mathrm{Mat}_n(\mathbb{C})$.
- The topological Brauer group, $\mathrm{Br}_{\mathrm{top}}(X)$.
- The degree of an Azumaya algebra.
- The topological period $\mathrm{per}_{\mathrm{top}}(\alpha)$
- The topological index $\mathrm{ind}_{\mathrm{top}}(\alpha)$.

There are some comparisons if X is a complex variety:

- An étale (ordinary) Azumaya algebra is a topological one.
- $\mathrm{Br}(X) \to \mathrm{Br}_{\mathrm{top}}(X)$
- $\mathrm{per}_{\mathrm{top}}(\alpha) | \mathrm{per}(\alpha)$
- $\mathrm{ind}_{\mathrm{top}}(\alpha) | \mathrm{ind}(\alpha)$.

Universal Topological Azumaya Algebras

For the objects we consider, there is a natural identification $\mathrm{Br}(X) = \mathrm{H}^2(X, \mathbb{G}_m)_{\mathrm{tors}}$.

Universal spaces, I

The space $\mathrm{B}\ \mathrm{PGL}_n$ is equipped with a universal topological Azumaya algebra of degree n.

The period, index are both n

Universal spaces, II

- Define $\mathrm{P}(n, an) = \mathrm{SL}_{an} / \mu_n$.
- Get maps $\mathrm{PGL}_n \to \mathrm{P}(n, an) \to \mathrm{PGL}_{an}$: block-summation and projection.
- $\mathrm{B}\ \mathrm{PGL}_n \to \mathrm{B}\ \mathrm{P}(n, an)$ yields $\mathrm{Br}_{\mathrm{top}} = \mathrm{H}^2(\cdot, \mathbb{G}_m) = \mathbb{Z}/n$.
- $\mathrm{B}\ \mathrm{P}(n, an)$ is equipped with a universal topological Azumaya algebra of degree an but period n.

Not algebraic

The objects $\mathrm{B}\ \mathrm{P}(n, an)$ and $\mathrm{B}\ \mathrm{PGL}_n$ are not algebraic. Instead we use Totaro-style approximations, which are varieties $X \to \mathrm{B}\ G$ which are m-weak-equivalences.

Lemma (Antieau-W.)

If A, A' are 5-weakly-equivalent to $\mathrm{B}\ \mathrm{PGL}_2$, then any map $A \to A'$ inducing an isomorphism on $\mathrm{Br}_{top} = \mathrm{H}^3(\cdot, \mathbb{Z})$ is a 2-local 5-weak-equivalence.

A 2-local 5-weak-equivalence is a map inducing an isomorphism on $\pi_i(\cdot) \otimes_{\mathbb{Z}} \mathbb{Z}_{(2)}$ for $0 \leq i \leq 5$.
Heavily influenced by a theorem of Jackowski, McClure & Oliver restricting $\mathrm{B}\ G \to \mathrm{B}\ G$ when G is a simple Lie group.

Theorem (Antieau-W., 2012)

There exists a ring R and a class $[\mathcal{A}]$ in $\mathrm{Br}(\mathrm{Spec}\ R)$ such that $\mathrm{ind}([\mathcal{A}]) = 2$, but such that there are no degree-2 Azumaya algebras over R equivalent to $\mathcal{A}$.

Proof of Main Theorem

Proof.

- Let Spec R be an affine, integral complex variety that is 5-weakly-equivalent to $\mathrm{B}\ \mathrm{P}(2, 2n)$ for some odd n.
- $\mathrm{Br}_{\mathrm{top}}(\mathrm{Spec}\, R) = \mathbb{Z}/2$, generated by the class of $\mathcal{A}$, of period 2, index 2 and degree $2n$.
- We can approximate the block-summation map $\mathrm{B}\ \mathrm{PGL}_2 \to \mathrm{B}\, P(2, 2n)$ by $X \to \mathrm{Spec}\, R$, which is 5-weakly-equivalent to the original map.
- Suppose we could find a Brauer-equivalent, degree-2 algebra. We'd find a map
$$X \longrightarrow \mathrm{Spec}\, R \longrightarrow \mathrm{B}\ \mathrm{PGL}_2$$
indcuing an isomorphism on $\mathrm{Br}_{\mathrm{top}}$.
- By the lemma, a 2-local 5-weak-equivalence.
- Impossible on the homotopy groups: $\pi_5(\mathrm{B}\ \mathrm{PGL}_2) = \mathbb{Z}/2$, $\pi_5(\mathrm{B}\, P(2, 2n)) = 0$.

References I

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