Math 149A Fall 2017 Final Exam Solutions (White version)

Note: This exam was graded out of 79 points, not the 80 originally stated on the first page (problem 8 was worth 8 points instead of 9).

1. (3 points each, 9 points total). Each of the following "theorems" is actually FALSE! Give a counterexample for each of them (e.g. for part a, give an example of a pair of random variables which has 0 correlation, but are not independent).

Note: For all of these, there were many possible solutions. I'm only giving one of them, but you didn't have to match it specifically

Part a: If X and Y are uncorrelated (have $\rho(X,Y)=0$), then X and Y are independent.

Solution: We had several examples both in class and on the last homework like this. For example, let (X,Y) be uniform on $\{(0,1),(0,-1),(1,0),(-1,0)\}$. Then we have $\mathbf{E}(X)=\mathbf{E}(Y)=\mathbf{E}(XY)=0$, so the variables are uncorrelated. But X's distribution changes based on Y: If $Y=1$, then $X=0$, while if $Y=0$, X is never 0.

Part b: If A and B are independent events, and B and C are independent events, then A and C are independent events.

Solution: I roll a die once. Let X be the number I rolled. Let A be the event "I roll an even number", B be the event "I roll a multiple of three", and C be the event "I roll an odd number". We have

$$\begin{aligned}\mathbf{P}(A)=\mathbf{P}(C)=\frac{1}{2} \quad &, \quad \mathbf{P}(B)\frac{2}{6}=\frac{1}{3} \\ \mathbf{P}(A\cap B) \quad &= \quad \mathbf{P}(X=6)=\frac{1}{6} \\ \mathbf{P}(B\cap C) \quad &= \quad \mathbf{P}(X=3)=\frac{1}{6}\end{aligned}$$

(e.g. $A\cap B$ means I rolled a number which is both even and a multiple of 3, which is only 6). In particular, we have

$$\mathbf{P}(A\cap B)=\mathbf{P}(A)\mathbf{P}(B),$$

so A and B are independent. Similarly, B and C are independent. But A and C are not independent: If I tell you I rolled an even number, that makes it rather less likely I rolled an odd number!

Part c: If A and B are two events, $\mathbf{P}(A|B)=\mathbf{P}(B|A)$.

Solution: Take A and B as in the previous example. We have

$$\begin{aligned} \mathbf{P}(A|B) &= \frac{\mathbf{P}(A\cap B)}{\mathbf{P}(B)} = \frac{1/6}{1/3} = \frac{1}{2} \\ \mathbf{P}(B|A) &= \frac{\mathbf{P}(A\cap B)}{\mathbf{P}(A)} = \frac{1/6}{1/2} = \frac{1}{3} \end{aligned}$$

(Note that since A and B are independent, it's not surprising that $\mathbf{P}(A|B)$ is the same thing as we got for $\mathbf{P}(A)$ in the previous part).

If you look at the formulas for $\mathbf{P}(A|B)$ and $\mathbf{P}(B|A)$, you can see that any events whatsoever with $\mathbf{P}(A) \neq \mathbf{P}(B)$ make a counterexample here.

Problem 2: (8 points total) A jar contains 12 balls, of which 3 are red, 4 are blue, and 5 are green. I draw 3 balls out of this jar, without replacement (all balls being equally likely).

Note: For all three parts, you can either have order matter or not matter. The important thing is to be consistent

Part a: (2 points) What is the sample space of this experiment, and what is its size?

Solution: The Sample space is the collection of all sets of three balls. The size is $\binom{12}{3}$ (if order doesn't matter) or $12 \times 11 \times 10$ (if order does matter)

Part b: (3 points) What is the probability that all three balls are the same?

Solution: If order doesn't matter: We have

$$\begin{aligned} \mathbf{P}(\text{all same color}) &= \mathbf{P}(\text{all red}) + \mathbf{P}(\text{all blue}) + \mathbf{P}(\text{all green}) \\ &= \frac{\binom{3}{3}}{\binom{12}{3}} + \frac{\binom{4}{3}}{\binom{12}{3}} + \frac{\binom{5}{3}}{\binom{12}{3}} \end{aligned}$$

If order does matter, the calculation is similar, but now we use permutations instead of combinations everywhere, getting

$$\mathbf{P}(\text{all same color}) = \frac{3 \times 2 \times 1 + 4 \times 3 \times 2 + 5 \times 4 \times 3}{12 \times 11 \times 10}$$

If you work out the numbers, both answers are the same.

Part c: (3 points) What is the probability that I draw one ball of each color?

Solution: If order doesn't matter: There are $(3)(4)(5) = 60$ ways of choosing one ball of each color. So the probability is

$$\frac{60}{\binom{12}{3}}$$

If you're saying order does matter, there's now $(3)(4)(5)(3!) = 360$ ways of picking the balls of each color (the extra 3! corresponds to how you now need to worry about what order you drew the three colors in), and the probability now becomes

$$\frac{360}{12 \times 11 \times 10},$$

which is again the same.

Note: The events in parts b and c are not complements! For example, you can have two red and one blue ball, in which case neither event happens. So you can't just take your answer in part b and subtract from 1.

3. (7 points total) I have two dice, one with 4 sides numbered $\{1, 2, 3, 4\}$, and the other with 6 sides numbered $\{1, 2, 3, 4, 5, 6\}$. Both are fair dice (equally likely to come up on any side).

Part a: (6 points) I choose a die at random (both dice being equally likely) and roll it twice, getting 1 then 4. Given the results of my two die rolls, what is the probability that I chose the die with 6 sides?

Solution: We use Bayes' Theorem. Let C_1 be the event I chose the 4 sided die, C_2 be the event I chose the 6 sided die, and A be the event I rolled $1, 4$. We have:

$$\begin{aligned} \mathbf{P}(C_1) = \mathbf{P}(C_2) &= \frac{1}{2} \\ \mathbf{P}(A|C_1) &= \left(\frac{1}{4}\right)^2 = \frac{1}{16} \\ \mathbf{P}(A|C_2) &= \left(\frac{1}{6}\right)^2 = \frac{1}{36} \end{aligned}$$

So by Bayes' theorem,

$$\mathbf{P}(C_2|A) = \frac{\mathbf{P}(A|C_2)\mathbf{P}(C_2)}{\mathbf{P}(A|C_1)\mathbf{P}(C_1) + \mathbf{P}(A|C_2)\mathbf{P}(C_2)} = \frac{(\frac{1}{2})(\frac{1}{36})}{(\frac{1}{2})(\frac{1}{16}) + (\frac{1}{2})(\frac{1}{36})} = \frac{4}{13}$$

Part b: (1 point) Suppose that instead my two rolls were 1, then 5. What would be the conditional probability I chose the die with 6 sides given these rolls?

Solution: Now $\mathbf{P}(A|C_1) = 0$. So you can either run through the Bayes theorem calculations again, or just recognize that since the 4 sided die is now impossible, the probability of the 6 sided die must be 1.

4. (7 points total) I have a biased coin that comes up heads with probability 2/3. I flip the coin 4 times (different flips are independent). Let X be the number of heads total I get in my four flips.

Part a (1 point) What is $\mathbf{E}(X)$?

Solution: X is Binomial with parameters $n = 4, p = \frac{2}{3}$. So $\mathbf{E}(X) = np = \frac{8}{3}$.

Part b (2 points) What is $\mathbf{Var}(3X)$?

Solution: We have $\mathbf{Var}(3X) = 9\mathbf{Var}(X)$, and $\mathbf{Var}(X) = np(1-p) = \frac{8}{9}$. So $\mathbf{Var}(3X) = 8$.

Part c (4 points) Let $F_X(x)$ be the cdf of X. What is the value of $F(2)$? You do not need to simplify your answer.

Solution: By definition, $F(2) = \mathbf{P}(X \leq 2)$. This is just

$$\mathbf{P}(X=0) + \mathbf{P}(X=1) + \mathbf{P}(X=2) = \binom{4}{0}\left(\frac{1}{3}\right)^4 + \binom{4}{1}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^1 + \binom{4}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2.$$

5. (10 points total) Let X be a random variable whose pdf has the form

$$f(x) = cx^2 \quad \text{for } -1 \le x \le 1$$

and 0 otherwise.

Part a: (2 points) Determine the value of c

Solution: For $f(x)$ to be a pdf, it must integrate out to 1. So we have

$$1 = \int_{-1}^{1} cx^2 \, dx = \frac{cx^3}{3}\bigg|_{-1}^{1} = \frac{2c}{3}$$

Solving, we get $c = \frac{3}{2}$.

If you did not solve part a, you may assume for the remaining two parts that $c = 2$ (this isn't the actual value of c, just a way for you to show you can still do this problem).

Part b: (4 points) Compute the variance of X.

Solution: We have

$$\mathbf{E}(X) = \int_0^1 \frac{3}{2}x^3 \, dx = \frac{3}{8}x^4\bigg|_{-1}^{1} = 0$$

(You could also have seen this by symmetry) and

$$\mathbf{E}(X^2) = \int_0^1 \frac{3}{2}x^5 \, dx = \frac{3}{10}x^4\bigg|_{-1}^{1} = \frac{3}{5}$$

So we have

$$\mathbf{Var}(X) = \mathbf{E}(X^2) - (\mathbf{E}(X))^2 = \frac{3}{5}$$

Part c: (4 points) Let $Y = 2X^3$. What is the pdf of Y?

Solution: We use the Jacobian technique (you could also have used cdfs). We have

$$X = (Y/2)^{1/3},$$

so

$$\frac{\partial X}{\partial Y} = (\frac{1}{3}(Y/2)^{-2/3})\frac{1}{2} = \frac{1}{6}(Y/2)^{-2/3}$$

(Chain rule) Substituting Y for X in the pdf and multiplying by the Jacobian, we get

$$f(y) = \left(\frac{3}{2}\left(\frac{y}{2}\right)^{2/3}\right))\left(\frac{1}{6}\left(\frac{y}{2}\right)^{-2/3}\right) = \frac{1}{4}$$

With new range $-2 \le y \le 2$ (plug the old range for X into $Y = 2X^3$)

6. (10 points total) Let X and Y be two random variables whose joint pdf is

$$f(x,y) = \begin{cases} 6xy^2 & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Part a: (2 points) Are X and Y independent? Explain. (A correct answer with no explanation is worth 0 points)

Solution: Yes. For example, the pdf factors as $g(x)h(y)$, where $g(x) = 6x$ for $0 \leq x \leq 1$ and $h(y) = y^2$ for $0 \leq y \leq 1$. (Recall question 1 on the last homework; it's important that the ranges factor too!)

Part b: (4 points) Compute the joint CDF of X and Y.

Solution: For $0 \leq x \leq 1$ and $0 \leq y \leq 1$ we have

$$\begin{aligned} F(x,y) &= \mathbf{P}(X \leq x \text{ and } Y \leq y) \\ &= \int_0^x \int_0^y 6st^2 \, dt \, ds \\ &= \int_0^x \left(2st^3\right)\Big|_{t=0}^{y} ds \\ &= \int_0^x 2sy^3 ds \\ &= s^2y^3\Big|_{s=0}^{x} = x^2y^3 \end{aligned}$$

Part c: (4 points) Let $Z = X + Y$, and let $g(z)$ be the pdf of Z. What is $G(1.5)$?

Solution: For convolution, we're supposed to integrate $g(z-y, y)\, dy = g(1.5-y, y)\, dy$. The tricky part is determining the bounds of integration. Here's two approaches:

Approach 1: Our pdf is only positive when both coordinates are between 0 and 1. In particular, y needs to be at least 0.5 (to make the first coordinate less than 1) and at most 1 (to make the second coordinate less than 1).

Approach 2: Our region is the unit square. The line $x + y = 1.5$ intersects the boundaries of the square at $(1, 0.5)$ and $(0.5, 1)$ (draw a picture!). Again this says y should be between 0.5 and 1.

Either way, you get

$$\begin{aligned} \int_{0.5}^1 6(1.5-y)y^2 \, dy &= \int_{0.5}^1 9y^2 - 6y^3 \, dy \\ &= 3y^3 - 1.5y^4\Big|_{0.5}^{1} \\ &= (3-1.5) - (3(0.5)^3 - 1.5(0.5)^4) \end{aligned}$$

This turns out to be equal to $39/32 \approx 1.22$, but you had no need to simplify.

7. (6 points total) Let X and Y be independent random variables, each of which is non-negative, and satisfying $\mathbf{E}(X) = 3$ $\mathbf{E}(Y) = 4$, $\mathbf{E}(X^2Y^2) = 150$

Part a: (4 points) Using Markov's inequality, give a bound on $\mathbf{P}(X \geq 12 \text{ and } Y \geq 12)$

Solution: Since X and Y are independent, we have

$$\mathbf{P}(X \geq 12 \text{ and } Y \geq 12) = \mathbf{P}(X \geq 12)\mathbf{P}(Y \geq 12)$$

By Markov

$$\mathbf{P}(X \geq 12) \leq \frac{\mathbf{E}(X)}{12} = \frac{1}{4}$$

and similarly

$$\mathbf{P}(Y \geq 12) \leq \frac{\mathbf{E}(Y)}{12} = \frac{1}{3}$$

Multiplying, we have

$$\mathbf{P}(X \geq 12 \text{ and } Y \geq 12) \leq \frac{1}{12}.$$

Part b: (2 points) Let $Z = XY$. What is the variance of Z?

Solution: We have

$$\mathbf{Var}(Z) = \mathbf{E}(Z^2) - \mathbf{E}(Z)^2$$

We're given $\mathbf{E}(Z^2) = 150$, and by independence we have $\mathbf{E}(XY) = \mathbf{E}(X)\mathbf{E}(Y) = 12$. So

$$\mathbf{Var}(Z) = 150 - 12^2 = 6$$

8. (9 points total) Let X and Y be two random variables with

$$\begin{array}{ll} \mathbf{P}((X,Y)=(0,1))=\frac{1}{3} & \mathbf{P}((X,Y)=(1,2))=\frac{1}{3} \\ \mathbf{P}((X,Y)=(0,0))=\frac{1}{6} & \mathbf{P}((X,Y)=(1,0))=\frac{1}{6} \end{array}$$

Part a: (4 points) Compute the conditional expectation $\mathbf{E}(Y|X=1)$ (the expectation of Y given that $X=1$).

Solution: We have $\mathbf{P}(X=1)=\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$. It follows that

$$\begin{aligned} \mathbf{P}(Y=2|X=1) &= \frac{1/3}{1/2}=\frac{2}{3} \\ \mathbf{P}(Y=0|X=1) &= \frac{1/6}{1/2}=\frac{1}{3} \end{aligned}$$

To find condition expectation, we use the usual expectation formula with these new conditional probabilities.

$$\mathbf{E}(Y|X=1)=2(2/3)+0(1/3)=4/3$$

Part b: (4 points) Compute the covariance of X and Y.

Solution: We have

$$\begin{aligned} \mathbf{E}(X) &= \frac{1}{3}(0)+\frac{1}{6}(0)+\frac{1}{3}(1)+\frac{1}{6}(1)=\frac{1}{2} \\ \mathbf{E}(Y) &= \frac{1}{3}(1)+\frac{1}{6}(0)+\frac{1}{3}(2)+\frac{1}{6}(0)=1 \\ \mathbf{E}(XY) &= \frac{1}{3}(0)(1)+\frac{1}{6}(0)(0)+\frac{1}{3}(1)(2)+\frac{1}{6}(1)(0)=\frac{2}{3} \end{aligned}$$

This means $\mathbf{Cov}(X,Y)=\mathbf{E}(XY)-\mathbf{E}(X)\mathbf{E}(Y)=\frac{2}{3}-\frac{1}{2}=\frac{1}{6}$.

9. (6 points) Let X_1 and X_2 be independent variables that have the same distribution and finite mean and variance.

$$\begin{aligned} Y_1 &= X_1 + X_2 \\ Y_2 &= X_1 - X_2 \end{aligned}$$

Explain why the correlation between Y_1 and Y_2 must be 0.

Solution: We have

$$\mathbf{E}(Y_1Y_2) = \mathbf{E}((X_1 + X_2)(X_1 - X_2)) = \mathbf{E}(X_1^2 - X_2^2) = \mathbf{E}(X_1^2) - \mathbf{E}(X_2^2)$$

by linearity of expectation. But if X_1 and X_2 have the same distribution, $\mathbf{E}(X_1^2) = \mathbf{E}(X_2^2)$. So we must have

$$\mathbf{E}(Y_1Y_2) = 0$$

Similarly,

$$\mathbf{E}(Y_2) = \mathbf{E}(X_1) - \mathbf{E}(X_2) = 0$$

Since X_1 and X_2 have the same expectation. This means that

$$\mathbf{Cov}(Y_1, Y_2) = \mathbf{E}(Y_1Y_2) - \mathbf{E}(Y_1)\mathbf{E}(Y_2) = 0 - \mathbf{E}(Y_1) \times 0 = 0$$

Since Y_1 and Y_2 have 0 covariance, they also have 0 correlation.

Note: The most common mistake here was to say either "X_1 and X_2 have the same distribution, so $X_1 - X_2 = 0$" or "Y_1 and Y_2 are independent, so must have 0 covariance". Neither of these is actually true. For example, suppose I roll two fair six sided dice, and let X_1 and X_2 be the numbers on the two dice. Then X_1 and X_2 have the same distribution (equally likely to be anything from 1 to 6), but whenever the dice show different numbers we have $X_1 - X_2 \neq 0$. Furthermore, Y_1 and Y_2 are *not* independent in this example: If $Y_1 = 12$, we know that both dice must have come up 6, meaning that Y_2 must be 0.

Optional Remark: You can generalize this problem, Suppose that we had a vector

$$X = (X_1, X_2, \ldots, X_n)$$

of random variables, where each X_i had the same variance σ^2 and the X_i were independent. If we take two vectors $v = (v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$ and let

$$\begin{aligned} Y_1 &= v \cdot x = v_1X_1 + v_2X_2 + \cdots + v_nX_n \\ Y_2 &= w \cdot x = w_1X_1 + w_2X_2 + \cdots + w_nX_n \end{aligned}$$

Then a bit of calculation gives

$$\mathbf{Cov}(Y_1, Y_2) = (v \cdot w)\sigma^2$$

and

$$\rho(Y_1, Y_2) = \frac{(v \cdot w)}{|v||w|},$$

which, if you remember your linear algebra, is the cosine of the angle between v and w.

In the exam question, v was $(1, 1)$ and w was $(1, -1)$. That made the vectors orthogonal, so the dot product is 0 (the angle between them is 90 degrees.

The intuition here: Y_1 and Y_2 in a sense correspond to looking at the projection of X in two different directions. If the directions are almost the same (the angle between the two are equal), then we expect the projections to be closely related to each other. But if the directions are far apart from each other, there's less of a relationship.

10. (8 points total) Let X, Y, Z be three random variables with

$$\begin{array}{lll} \mathbf{E}(X) = 4 & \mathbf{E}(Y) = 7 & \mathbf{E}(Z) = 11 \\ \mathbf{Var}(X) = 4 & \mathbf{Var}(Y) = 7 & \mathbf{Var}(Z) = 9 \\ \mathbf{Cov}(X,Y) = -2 & \mathbf{Cov}(X,Z) = -3 & \mathbf{Cov}(Y,Z) = -5 \end{array}$$

Part a: (3 points) What is $\mathbf{E}(X^2 - 2Y)$?

Solution: We have $\mathbf{E}(X^2) = \mathbf{E}(X)^2 + \mathbf{Var}(X) = 20$ and $\mathbf{E}(-2Y) = -2\mathbf{E}(Y) = -14$. By linearity of Expectation, $\mathbf{E}(X^2 - 2Y) = 20 - 14 = 6$.

Part b: (3 points) Let $W = X + Y + Z$. What is the variance of W?

Solution: We have

$$\mathbf{Var}(W) = \mathbf{Var}(X) + \mathbf{Var}(Y) + \mathbf{Var}(Z) + 2(\mathbf{Cov}(X,Y) + \mathbf{Cov}(X,Z) + \mathbf{Cov}(Y,Z))$$

Plugging everything in, we get $\mathbf{Var}(W) = 0$.

Part c: (2 points) Given your answer to part b, what can you conclude about the distribution of W?

Solution: Since the variance is 0, we know that W must be a constant (equal to a single value with probability 1). Since

$$\mathbf{E}(W) = \mathbf{E}(X) + \mathbf{E}(Y) + \mathbf{E}(Z) = 22,$$

that must be the constant that W equals.