Math 149A HW 2
Solutions

1. For each of the following functions, determine whether or not it is an actual probability function. In particular, you should either explain why all three of the properties in the definition are true (i.e. explain why it's a probability function), or show that one of the three properties is NOT true (which would show that it is not a probability function).

Part a: The sample space is $\{0,1\}$. We have $P(\emptyset) = 0$, $P(0) = 0.6$, $P(1) = 0.3$, and $P(\{0,1\}) = 1$.
Part b: The sample space is $[0,1]$. We have $P(S) = 1$ if S contains 1, and $P(S) = 0$ if S does not contain 1.
Part c: The sample space is $[-2,4]$. The function P assigns to a set S the value $\int_S \frac{1}{6}x^2 - \frac{1}{3}dx$

Solution: The first function **is not** a probability function. The sets $\{0\}$ and $\{1\}$ are disjoint, but $P(\{0,1\}) = 1 \neq P(\{0\}) + P(\{1\})$.

The part b function **is** a probability function. The first property is true because all probabilities are either 0 or 1. The second property is true because if $\mathscr{C}$ is the entire sample space, then $\mathscr{C}$ contains 1, so $P(\mathscr{C}) = 1$. The third property is true because if A and B are two disjoint sets, then either

- Neither contains 1, so $P(A \cup B) = P(A) + P(B) = 0$
- One set contains 1. Then that set has probability 1, the other set has probability 0 (since the sets are disjoint, they can't both contain 1), and the union has probability 1 (it contains 1). So we have $P(A \cup B) = P(A) + P(B) = 1$

The same argument works when you consider more than two sets.

The part c function **is not** a probability function for multiple reasons (you only needed to give one). One problem is that the probability of the whole space is

$$\int_{-2}^{4} \frac{1}{6}x^2 - \frac{1}{3}\,dx = 2.$$

Another is that, because $\frac{1}{6}x^2 - \frac{1}{3}$ is negative in places, the function gives certain sets a negative "probability". For example, the probability of $[0,1]$ is

$$\int_0^1 \frac{1}{6}x^2 - \frac{1}{3}\,dx = -\frac{5}{18}$$

2. (based on Exercise 1.3.5 from the text). Suppose that our sample space is $\mathscr{C} = [0, \infty)$ For a set S, we define

$$\mathbf{P}(C) = \int_C e^{-x}\, dx$$

As discussed in class, this sort of function will always satisfy the third part of the definition of probability (probability of disjoint unions) just by the rules of calculus.

Part a: Explain why $P(S) \geq 0$ for any S. (i.e. what specific property of e^{-x} makes this true?)

Solution: The important thing about e^{-x} here is that it's always non-negative. So when I integrate it over an interval, I'll always get a non-negative number.

Part b: Check if the remaining part of the definition of probability (that the probability of the whole sample space is 1) holds.

Solution: This is a direct calculation. We have

$$\begin{aligned}
\mathbf{P}(\mathscr{C}) &= \int_0^\infty e^{-x}\, dx \\
&= \lim_{t\to\infty} \int_0^t e^{-x}\, dx \\
&= \lim_{t\to\infty} -e^{-x}|_0^t \\
&= \lim_{t\to\infty} 1 - e^{-t} = 1
\end{aligned}$$

Part c: Let $S = [4, \infty)$. What is $\mathbf{P}(S)$? What is $\mathbf{P}(S^C)$?

Solution: Since $S^C = [0, 4)$, we have

$$\mathbf{P}(S^C) = \int_0^4 e^{-x}\, dx = -e^{-x}|_0^4 = 1 - e^{-4}$$

This implies that

$$\mathbf{P}(S) = 1 - (1 - e^{-4}) = e^{-4}$$

Remark: You could also have computed $\mathbf{P}(S)$ directly first, then used that to get $\mathbf{P}(S^C)$. Or you could have integrated twice.

3. If the sample space is $\mathscr{C} = C_1 \cup C_2$ and if $\mathbf{P}(C_1) = 0.8$ and $\mathbf{P}(C_2) = 0.5$, find $\mathbf{P}(C_1 \cap C_2)$.

Solution: By inclusion-exclusion, we have $P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2)$. In this case the union has probability 1 (its the whole sample space), and this implies $1 = 0.8 + 0.5 - P(C_1 \cap C_2)$. So the intersection has probability 0.3.

4. Let C_1, C_2, and C_3 be three mutually disjoint subsets of the sample space $\mathscr{C}$. Find $\mathbf{P}[(C_1 \cup C_2) \cap C_3]$ and $\mathbf{P}(C_1^c \cup C_2^c)$.

Solution: This is an exercise in untangling what the sets are. Since C_1, C_2, and C_3 are mutually disjoint, then there are no elements in $(C_1 \cup C_2) \cap C_3$ (such an element would be in both C_3 and one of C_1 and C_2). So $P[(C_1 \cup C_2) \cap C_3] = 0$.

On the other hand, $C_1^c \cup C_2^c$ is the entire sample space. Anything not in C_1^c (in other words, anything in C_1) must be in C_2^c (since nothing is in both C_1 and C_2). So $P(C_1^c \cup C_2^c) = 1$.

5. A person has purchased 10 of 1000 tickets sold in a certain raffle. To determine the five prize winners, five tickets are to be drawn at random and without replacement. Compute the probability that this person wins at least one prize. *Hint:* First compute the probability that the person does not win a prize.

Solution: This is an analogue of the "Alice and Bob" warmup from lecture last week. We start by computing the probability the person does *not* win a prize.

There's $\binom{1000}{5}$ ways to draw the prize winners, and $\binom{990}{5}$ ways to draw 5 prize winners that don't include the person. So the probability they don't win is

$$\frac{\binom{990}{5}}{\binom{1000}{5}}$$

Where we need to be a bit careful here is entering things in the calculator, or some calculators, at least. If you try and type something like

$$\frac{1000!}{995!5!},$$

you may get an error message, because 1000! is enormous. So instead, we'll do a bit of manipulation before using a calculator, writing

$$\begin{aligned}
\frac{\binom{990}{5}}{\binom{1000}{5}} &= \frac{\frac{(990)(989)(988)(987)(986)}{5!}}{\frac{(1000)(999)(998)(997)(996)}{5!}} \\
&= \frac{(990)(989)(988)(987)(986)}{(1000)(999)(998)(997)(996)} \\
&\approx 0.9508
\end{aligned}$$

Subtracting, the probability they win at least one prize is $1 - 0.9508 \approx 0.0492$.

6. In a lot of 50 light bulbs, there are 2 bad bulbs. An inspector examines five bulbs, which are selected at random and without replacement.

Solution: This is similar to the previous problem. We start by computing the probability of no defective bulbs, which is

$$\frac{\binom{48}{5}}{\binom{50}{5}}$$

for the exact same reasons as before. Either doing some fancy footwork like we did before or just entering things in our calculator (the numbers are smaller now, so overflow isn't as much of a problem), we get 0.808. So the probability of detecting a bad bulb is $1 - 0.808 = 0.192$.

For part b, we're essentially asking "how large should k be so that picking k bulbs instead of 5 leads to a detection probability of at least 0.5? This means the probability we fail to find a bad bulb is at most 0.5. That probability is

$$\frac{\binom{48}{k}}{\binom{50}{k}}$$

At this point one option is just to try k until we get one that works. Another option is to do a bit of manipulation, writing the probability as

$$\frac{\frac{48!}{k!(48-k)!}}{\frac{50!}{k!(50-k)!}} = \frac{\frac{48!}{(48-k)!}}{\frac{50!}{(50-k)!}} = \frac{48!}{50!}\frac{(50-k)!}{(48-k)!}$$

Now, $50! = 50 \times 49 \times 48 \times \cdots \times 1 = 50 \times 49 \times 48!$. So we can write

$$\frac{48!}{50!} = \frac{1}{(50)(49)}$$

Similarly, we have

$$\frac{(50-k)!}{(48-k)!} = (50-k)(49-k)$$

So the entire expression is

$$\frac{(50-k)(49-k)}{2450}$$

We want this to be at most $\frac{1}{2}$, which is equivalent to $(50-k)(49-k)$ being at most $\frac{2450}{2} = 1225$. Direct calculation (or solving for k) gives that $k = 14$ is not enough, but $k = 15$ is.