1. Let $X_1, X_2, \ldots, X_{100}$ be independent variables, each drawn from a Gamma distribution with $\alpha = 2$ and $\beta = 4$. What is the approximate probability that $7 \le \overline{X} \le 9$)?

Solution: We have $\mu = E(X) = \alpha\beta = 8$, and $\sigma^2 = Var(X) = \alpha\beta^2 = 32$. It follows from the Central Limit Theorem that if

$$Z = \left(\overline{X} - 8\right) \frac{\sqrt{100}}{\sqrt{32}} \approx \left(\overline{X} - 8\right) 1.7678,$$

then Z is approximately equal to a standard $N(0,1)$ variable. In particular, we have

$$\begin{aligned}
P(7 \le \overline{X} \le 9) &= P(-1.7678 \le (\overline{X} - 8)(1.7678) \le 1.7678) \\
&\approx \Phi(1.7678) - \Phi(-1.7678) \\
&= \Phi(1.7678) - (1 - \Phi(1.7678)) \\
&\approx 0.9616 - (1 - 0.9616) \approx 0.923
\end{aligned}$$

2. Let $X_1, \ldots, X_{100}$ be independent variables, each with a $\chi^2(50)$ distribution. What, approximately, i sthe probability that their average is between 49 and 51?

Solution: We have $\mu = E(X) = 50$, and $\sigma^2 = Var(X) = 100$. It follows from the Central Limit Theorem that if

$$Z = \left(\overline{X} - 50\right) \frac{\sqrt{100}}{10} = \overline{X} - 50,$$

then Z is approximately equal to a standard $N(0,1)$ variable. In particular, we have

$$\begin{aligned}
P(49 \le \overline{X} \le 51) &= P(-1 \le \overline{X} - 50) \le 1) \\
&\approx \Phi(1) - \Phi(-1) \\
&\approx 0.8413 - (1 - 0.8413) \approx 0.683
\end{aligned}$$

3. Let Y be Binomial with parameters $n = 72$ and $p = 1/3$. Compute, approximately, the probability that $22 \le Y \le 28$, using a continuity correction if appropriate.

Solution: Here a continuity correction **is** appropriate, since we're approximating an integer-valued Binomial distribution by a continuous Normal distribution. Think of X as $X_1 + \cdots + X_{72}$, where each X_i is independently 1 with probability 1/3 and 0 with probability 2/3. We have

$$\begin{aligned}
E(X_i) &= (1/3)(1) + (2/3)(0) = 1/3 \\
E(X_i^2) &= (1/3)(1^2) + (2/3)(0^2) = 1/3 \\
Var(X_i) &= 1/3 - (1/3)^2 = 2/9
\end{aligned}$$

It follows from the Central Limit Theorem (sum form), that if

$$Z = \frac{Y - (72)(1/3)}{\sqrt{2/9}\sqrt{72}} = \frac{Y - 24}{4},$$

then Z is approximately equal to a standard $N(0,1)$. In particular, we have

$$\begin{aligned} P(22 \leq Y \leq 28) &= P(21.5 \leq Y \leq 28.5) \text{(the continuity correction)} \\ &= P(-0.625 \leq \frac{Y-24}{4} \leq 1.125) \\ &= \Phi(1.125) - \Phi(-0.625) \\ &= \Phi(1.125) - (1 - \Phi(0.625)) \\ &\approx 0.87 - (1 - 0.73) = 0.60 \end{aligned}$$

Note that we did **not** use a continuity correction on the first two problems – the Gamma and Chi-Square distributions are already continuous, so it doesn't apply.

4. Bob flips 60 fair coins and Alice flips 50 coins. Let B be the number of heads Bob gets, and A be the number of heads Alice gets.

Part a: If I were to approximate B by a normal distribution, what would be the mean and variance of that distribution? Repeat for Alice.

Solution: We have $E(B) = (60)(1/2) = 30$ and $Var(B) = (60)(1/2)(1 - 1/2) = 15$. When we approximate B by a normal distribution using the CLT, the approximating distribution has the same mean and variance. So we're approximating B by $N(30, 15)$. Similarly, we're approximating A by $N(25, 12.5)$.

Part b: As suggested by the hint, if B is approximately normal and A is approximately normal, than $B - A$ is also approximately normal, with mean $E(B) - E(A) = 5$ and Variance equal to $Var(B) + Var(A) = 27.5$ (convince yourself that that plus sign isn't a typo!). So if $X = B - A$, then

$$Z = \frac{X - 5}{\sqrt{27.5}} \approx N(0,1)$$

The probability that Alice gets at least as many heads as Bob is the probability that $X \leq 0$ (saying $B \leq A$ is equivalent to saying $B - A$ is at most 0). We have

$$\begin{aligned} P(X \leq 0) &= P(X \leq 0.5) \text{ (continuity correction)} \\ &= P(\frac{X-5}{\sqrt{27.5}} \leq \frac{0.5-5}{\sqrt{27.5}}) \\ &\approx P(Z \leq -0.8581) \\ &= \Phi(-0.8581) = 1 - \Phi(0.8581) \approx 0.195 \end{aligned}$$