Chow ring of heavy/light Hassett spaces

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joint with Siddarth Kannan and Shiyue Li

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- $\overline{\mathcal{M}}_{g,w}$ (Hassett) compactification: Moduli stack of nodal curves of arithmetic genus g with n weighted marked points, with weight vector $w = (w_1, \ldots, w_n)$.
  In more detail, it is the space of pairs (C, D) where C is a genus-g nodal curve and $D = \sum w_i P_i$ is a $\mathbf{Q}$-divisor weighted by w such that $K_C + D$ is ample along each irreducible component of C.

Special cases

- If D is reduced $\overline{\mathcal{M}}_{g,w} \cong \overline{\mathcal{M}}_{g,n}$, and $\overline{M}_{g,w} \cong \overline{M}_{g,n}$.

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$$\overline{M}_{0,w} = \overline{L}_n,$$

  the Losev-Manin compactification of $M_{0,n}$.
- We say w is *heavy/light* when

$$w = (1^m, \epsilon^{n-m}),$$

  where $\epsilon < \dfrac{1}{n-m}$ and $m \geq 2$.

Definition

Fix $k = \bar{k}$ and weights $w \in (\mathbf{Q} \cap (0,1])^n$ satisfying $\sum w_i > 2$. For a k-scheme B, a *family of nodal w-stable curves of genus 0 over B* is a flat proper morphism $\pi : \mathcal{C} \to B$, together with n sections $s_1, \ldots, s_n$, such that

(a) each geometric fiber $\mathcal{C}_b$ of π is a reduced, connected nodal curve of arithmetic genus 0;

(b) $s_{i_1}(b) = s_{i_2}(b) = \cdots = s_{i_p}(b)$ implies that $\sum_{j=1}^{p} w_{i_j} \leq 1$;

(c) for an irreducible component $T \subseteq \mathcal{C}_b$, we have

$$\#\{\text{nodes on } T\} + \sum_{j \in J(T)} w_j > 2,$$

where $J(T) = \{j : s_j(b) \in T\}$.

A curve appearing as the fiber $\mathcal{C}_b$ above is called a *w-stable tree of $\mathbf{P}^1$'s.*

Moduli functor

If we let $\overline{\mathcal{M}}_{0,w} : \mathsf{Sch}_k^{\mathsf{op}} \to \mathsf{Set}$ be the functor which takes a k-scheme B to the set of w-stable curves over B, then $\overline{\mathcal{M}}_{0,w}$ is represented by a smooth projective scheme $\overline{M}_{0,w}$.

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Example

When $w = (1^n)$, we recover the space $\overline{M}_{0,n}$ of n-marked stable rational curves, whose intersection theory was computed Keel, and then by Tavakol; both use a description of $\overline{M}_{0,n}$ as an iterated blowup of $(\mathbf{P}^1)^{n-3}$.

Theorem (Keel)

Let $n \geq 4$. The Chow ring of $\overline{M}_{0,n}$ is given as

$$A^*(\overline{M}_{0,n}) \cong \frac{\mathbf{Z}[D^S \mid S \subseteq \{1,\ldots,n\}, |S|, |S^c| \geq 2]}{\langle \textit{the following relations} \rangle},$$

1. $D^T - D^{T^c} = 0$ *for all* T.
2. $D^T D^S = 0$ *unless one of the following holds:*
$$S \subseteq T,\ T \subseteq S,\ S^c \subseteq T,\ T^c \subseteq S.$$
3. *For any four distinct elements* $i, j, k, \ell \in \{1,\ldots,n\}$,
$$\sum_{\substack{i,j \in S \\ k,\ell \notin S}} D^S = \sum_{\substack{i,k \in S \\ j,\ell \notin S}} D^S = \sum_{\substack{i,\ell \in S \\ j,k \notin S}} D^S.$$

$A^*(\overline{M}_{0,w})$ for heavy/light w

Theorem (Kannan, K., Li)

Let $m \geq 2$ and $n \geq 4$, and suppose that w is a heavy/light weight vector, with m heavy and $(n-m)$ light weights. Then the Chow ring of $\overline{M}_{0,w}$ is given as follows:

$$A^*(\overline{M}_{0,w}) = \frac{\mathbf{Z}\left[D^S \mid S \subsetneq \{2,\ldots,n\}, \sum_{i\in S} w_i > 1\right]}{\langle \textit{the following relations} \rangle},$$

1. $D^S D^T = 0$ *unless one of the following holds:* $S \subseteq T$, $T \subseteq S$, $S \cap T = \varnothing$.
2. *For any pair of two-element subsets* $\{i,j\}, \{k,\ell\} \subseteq \{2,\ldots,n\}$ *with* $i,k \leq m$, *we have the linear relation*

$$\sum_{\substack{S \not\supseteq \{k,\ell\} \\ S \supseteq \{i,j\}}} D^S = \sum_{\substack{S \supseteq \{k,\ell\} \\ S \not\supseteq \{i,j\}}} D^S.$$

Hyperplane arrangements

(We follow Maclagan-Sturmfels.) Let $\mathcal{A} = \{H_0, \ldots, H_n\}$ be an arrangement of $n+1$ hyperplanes in $\mathbf{P}^d$.

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Proposition

If $\mathbf{b}_0, \ldots, \mathbf{b}_n$ span k^{d+1}, then the hyperplane arrangment complement $X = \mathbf{P}^d \setminus \cup \mathcal{A}$ is isomorphic to a subvariety of the torus $T^n \subset \mathbf{P}^n$.

PROOF: Let B be a matrix with columns $\mathbf{b}_i$; it is $(d+1) \times (n+1)$.

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$$f_i=\sum_{j=0}^{n} a_{ij}x_j,$$

and define $I=\langle f_i \mid 1\leq i\leq n-d\rangle\subset k[x_0^{\pm 1},\dots,x_n^{\pm 1}]$. Now define

$$\beta : X\to V(I)$$

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by $\beta(z) = (\mathbf{b}_o \cdot z : \cdots : \mathbf{b}_n \cdot z) \in T^n$. Given $x \in V(I)$, $x \in \ker A$, so $x = B^T z$ for a unique $z \in X$. This provides an inverse to β. $\square$

Remark

Note that I is generated by linear forms. Reversing the above construction shows that every ideal generated by linear forms corresponds to a hyperplane arrangement.

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Definition

The *support* of $L = \sum a_i x_i \in I$ is $\{i \mid a_i \neq 0\}$. A nonempty subset $C \subseteq \{0, \ldots, n\}$ is a *circuit* of I if $C = \mathrm{supp}(L)$ for some nonzero linear form, and C is inclusion minimal with this property.

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Remark

C is a minimal linearly dependent set of columns of B, or when $\cap_{i \in C} H_i$ has codimension $|C| - 1$ and for all $j \in C$,

$$\cap_{i \in C} H_i = \cap_{i \in C, i \neq j} H_i.$$

Hyperplane arrangements

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Definition

The *order complex* of a poset is a simplicial complex whose vertices are the elements of the poset and whose simplices are all proper chains (not using $\{0\}, k^{d+1}$).

Matroids

Definition

A *matroid* is a pair $(E, \mathcal{C})$ where E is a finite set and $\mathcal{C}$ is a collection of nonempty subets of E, called the circuits of M, such that

1. No proper subset of a circuit is a circuit.
2. If C_1 and C_2 are distinct circuits and $e \in C_1 \cap C_2$, then $(C_1 \cup C_2) \setminus \{e\}$ contains a circuit.

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Remark

Equivalently, a *matroid* M is a pair (E, I) where E is a finite set and I is a collection of subsets of E satisfying

(i) $\varnothing \in I$,

(ii) if $X \in I$ and $Y \subseteq X$, then $Y \in I$, and

(iii) if $X, Y \in I$ with $|X| > |Y|$, then there exists $e \in X \setminus Y$ such that $Y \cup \{e\} \in I$.

Matroids

Example

If V is a vector space, and we have a finite collection of vectors $E = \{v_1, \ldots, v_n\} \subseteq V$, we can take the collection I of independent sets of subsets to be linearly independent subsets of E. Matroids which arise in this way are said to be *realizable*.

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Definition

The *closure* of $S \subseteq E$ in the matroid $M = (E, I)$ is

$$\mathrm{cl}(S) = \{x \in E \mid r(S \cup \{x\}) = r(S)\}.$$

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Given a finite graph G, the *graphic matroid* $M(G)$ associated to G has as its ground set the edge set $E(G)$ of G, where a subset S of $E(G)$ is independent if and only if S is the set of edges of some forest of G.

Remark

Given a finite graph G with vertex set $V(G) = \{v_1, \dots, v_n\}$, there is an associated hyperplane arrangement $\mathcal{A}_G$ in $\mathbf{C}^n$. Giving $\mathbf{C}^n$ coordinates $z_1, \dots, z_n$, we let H_{ij} be the hyperplane defined by $z_i - z_j = 0$. Then we may define

$$\mathcal{A}_G := \{H_{ij} \mid i, j \text{ are connected by an edge in } G\}.$$

Then the realizable matroid on the normal vectors of the hyperplanes in $\mathcal{A}_G$ is the same as the graphic matroid $M(G)$: linearly independent subsets correspond exactly to forests of G.

Definition

Given a finite matroid M on the ground set E, the *Bergman fan* $B(M)$ is a polyhedral cone complex that coincides with the order complex of the lattice of flats in M. More precisely, given a chain of flats

$$F_\bullet = \varnothing \subsetneq F_1 \subsetneq \cdots \subsetneq F_{r-1} \subsetneq F_r = E$$

where r is the rank of M and F_i is a flat of rank i in M for all $i = 1, \ldots, r$, a top-dimensional cone in the Bergman fan $\Sigma(M)$ is a cone in $\mathbf{R}^{|E|}$ spanned by rays corresponding to $v_{F_i} = -\sum_{j \in F_i} e_j$ for all $i = 1, \ldots, r-1$. The polyhedral structure thus obtained is called the *chain-of-flats subdivision* of the Bergman fan.

Let $\mathcal{F}$ denote the lattice of flats of a matroid M, and given two flats $F, F' \in \mathcal{F}$, write $[F, F'] := \{G \in \mathcal{F} \mid F \subseteq G \subseteq F'\}$. A *building set* for $\mathcal{F}$ is a subset $\mathcal{G}$ of $\mathcal{F} \setminus \{\varnothing\}$ such that the following holds: For any $F \in \mathcal{F} \setminus \{\varnothing\}$, let $G_1, \ldots, G_k$ be the maximal elements of $\mathcal{G}$ contained in F. Then there is an isomorphism of partially ordered sets:

$$\varphi_F : \prod_{j=1}^{k} [\varnothing, G_j] \to [\varnothing, F],$$

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where the jth component of φ_F is the inclusion $[\varnothing, G_j] \subseteq [\varnothing, F]$. A subset $\mathcal{S}$ of a building set is called *nested*, if for any set of incomparable elements $F_1, \ldots, F_l$ in $\mathcal{S}$ with $l \geq 2$, the join, i.e., the least upper bound, $F_1 \vee \ldots \vee F_l$ is not an element of $\mathcal{G}$.

The graphic matroid $M(w)$ of the reduced weight graph $G(w)$

Suppose $w = (1^{(m)}, \epsilon^{(n-m)})$ is a heavy/light weight vector. The *reduced weight graph* $G(w)$ of w has vertices $\{2, \ldots, n\}$, where vertex i is connected to vertex j if and only if $w_i + w_j > 1$.

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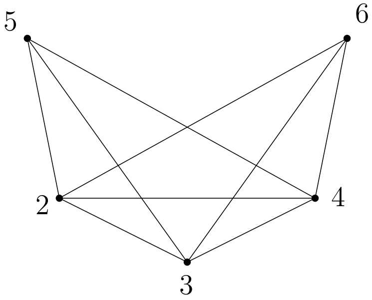
Example



The reduced weight graph $G(w)$ of $w = (1^{(4)}, \epsilon^{(2)})$

Combinatorial types of w-stable trees

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Let $w \in (\mathbf{Q} \cap (0,1])^n$, and suppose T is a w-stable tree. Then the *combinatorial type*, or *dual graph*, of T is a graph that has a vertex for each irreducible component of T, an edge between two vertices when the corresponding components share a node, and labelled rays recording the distribution of the marked points across the components.

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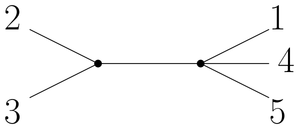
Example



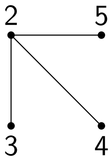
The combinatorial type of $(1^2, \epsilon^3)$-stable tree corresponding to the subset $S = \{2, 3\}$.

Theorem (Cavalieri-Hampe-Markwig-Ranganathan)

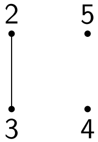
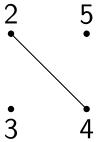
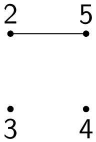
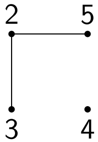
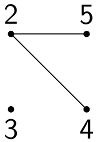
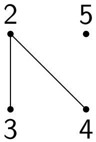
The set $\mathcal{G}$ of 1-connected flats of the graphic matroid $M(w)$ forms a building set for the lattice of flats of $M(w)$, and the nested-sets subdivision of $B'(M(w))$ with respect to the building set $\mathcal{G}$ embeds the cone complex $M_{0,w}^{trop}$ as a balanced fan.

The Losev-Manin space $\overline{M}_{0,w}$ with $w = (1, 1, \epsilon, \epsilon, \epsilon)$

The reduced weight graph $G(w)$ is the three-edge star:



1-connected flats of $M(w)$

rank 1	F_1	F_2	F_3
			
rank 2	F_4	F_5	F_6
			

Now we have the lattice of flats of $M(w)$ that contains six chains of flats

$$\varnothing \subsetneq F_i \subsetneq F_j \subsetneq E,$$

such as $\varnothing \subsetneq F_1 \subsetneq F_6 \subsetneq E$. These chains correspond to six top-dimensional cones C_{ij} spanned by v_{F_i} and v_{F_j}.

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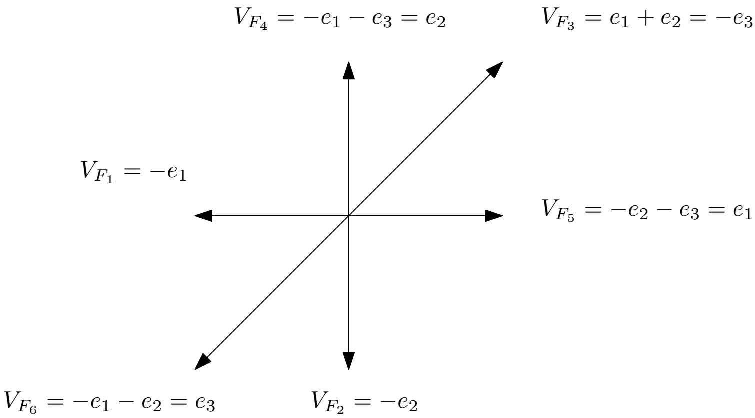
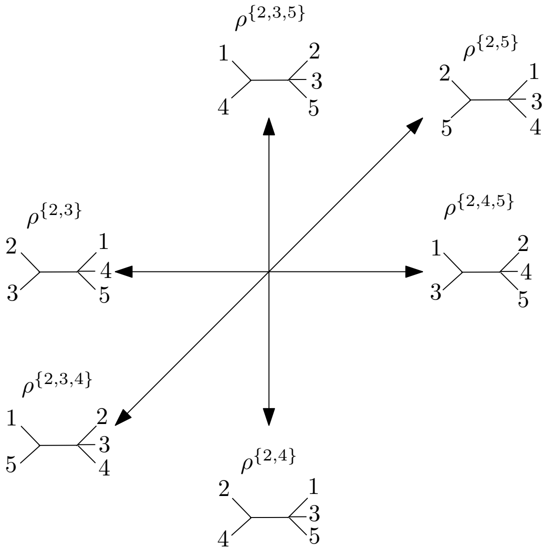
such as $\varnothing \subsetneq F_1 \subsetneq F_6 \subsetneq E$. These chains correspond to six top-dimensional cones C_{ij} spanned by v_{F_i} and v_{F_j}. Assigning the basis elements $-e_k$ to the rank-one flats F_k for all $k = 1, \ldots, |E|$, and modding out the relation that $e_1 + e_2 + e_3 = 0$, we obtain the reduced Bergman fan Σ_w embedded in $\mathbf{R}^2$.

Bergman fan



Bergman fan with w-stable trees

We have the presentation of Σ_w using w-stable trees.



By the Orbit-Cone Correspondence of toric divisor theory, 1-dimensional rays ρ^T of Σ_w for $T \subseteq \{2, \ldots, n\}$ correspond to the divisors D^T in $X(\Sigma_w)$.
Furthermore, the relations in Theorem 0.3 (1) are equivalent to saying that $D^S D^T = 0$ whenever ρ^S and ρ^T do not span a 2-dimensional cone in Σ_w.

Reading off the nine pairs of non-adjacent one-dimensional rays, we obtain the following Stanley-Reisner relations:

$$D^{\{2,3\}}D^{\{2,4\}} = D^{\{2,3\}}D^{\{2,4,5\}} = D^{\{2,3\}}D^{\{2,5\}} = 0,$$
$$D^{\{2,3,5\}}D^{\{2,4,5\}} = D^{\{2,3,5\}}D^{\{2,4\}} = D^{\{2,3,5\}}D^{\{2,3,4\}} = 0,$$
$$D^{\{2,5\}}D^{\{2,4\}} = D^{\{2,5\}}D^{\{2,3,4\}} = D^{\{2,5\}}D^{\{2,3\}} = 0.$$

The linear relations in the main theorem are

$$D^{\{2,3\}} + D^{\{2,3,5\}} = D^{\{2,4\}} + D^{\{2,4,5\}},$$
$$D^{\{2,3\}} + D^{\{2,3,4\}} = D^{\{2,5\}} + D^{\{2,4,5\}},$$
$$D^{\{2,4\}} + D^{\{2,3,4\}} = D^{\{2,5\}} + D^{\{2,3,5\}}.$$

Simplification

Therefore, the Chow ring of $\overline{M}_{0,w}$ is

$$A^*(\overline{M}_{0,w}) \cong \frac{\mathbf{Z}[D^S \mid S \subsetneq \{2,\dots,5\}, \sum_{i\in S} w_i > 1]}{\langle \text{Stanley-Reisner relations and linear relations} \rangle}.$$

Note that this coincides with the standard presentation of the Chow ring of $\mathbf{P}^2$ blown up at three torus-invariant points.