# A review of the theory of varieties

Javier González Anaya

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javiergo@ucr.edu

# Affine algebraic varieties

An **algebraic variety** is a geometric object that *locally* resembles the zero locus of a collection of polynomials.

We [redacted] consider[redacted] this local picture in detail.

## Definition

An **affine algebraic variety** is the common zero set of a collection $\{F_i\}_{i \in I}$ of complex polynomials on $\mathbb{C}^n$.

We write $V(\{F_i\}_{i \in I}) \subseteq \mathbb{C}^n$ for this set of common zeros.

# Examples

Here are some important examples of affine algebraic varieties:

- $V(0) = \mathbb{C}^n$;
- $V(1) = \emptyset$;
- $V(x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n) = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$;
- $V(x_1, x_2)$ is the $z$-axis in $\mathbb{C}^3$.

## Remarks

The variety $\mathbb{C}^n$ is usually referred to as *affine space*.

In geometry it's common to refer to $\mathbb{C}$ as the (complex) affine line, instead of complex plane! Similarly, we'll refer to $\mathbb{C}^2$ as the complex affine plane.

Affine varieties are closed subsets of $\mathbb{C}^n$ in the Euclidean topology. Indeed, polynomials are continuous functions from $\mathbb{C}^n$ to $\mathbb{C}$. Their zero locus is the inverse image of $0 \in \mathbb{C}$.

# The Zariski topology

Two ways to do algebraic geometry: studying holomorphic manifolds with the Euclidean topology, or taking the algebraic path. The starting point of the latter is the Zariski topology. (These two approaches are not independent, they're connected through a series of results by Serre that go under the blanket name "GAGA".)

Let's motivate the Zariski topology.

Consider two collections of polynomials $S = \{F_i\}$ and $T = \{G_j\}$.

- First note that
$$V(S) \cap V(T) = V(S \cup T).$$
- Similarly,
$$V(S) \cup V(T) = V(ST),$$
where $ST = \{F_i G_j : i \in I \text{ and } j \in J\}$.

# Example: The twisted cubic

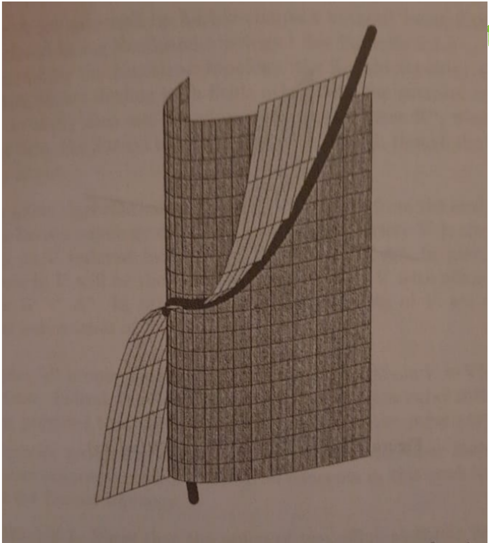


Figure 1.5. $V = \mathbb{V}(x^2 - y, x^3 - z) = \mathbb{V}(x^2 - y) \cap \mathbb{V}(x^3 - z)$.

K. Smith et al.
"Invitation to A.G."

# Definition

By the two previous observations we find the following:

- The intersection of arbitrarily many affine varieties is an affine variety.
- The union of finitely many affine varieties is an affine variety.
- The whole space $\mathbb{C}^n$ is an affine variety.
- The empty set is an affine variety.

Closed sets of a topology

## The Zariski topology on $\mathbb{C}^n$

The Zariski topology on $\mathbb{C}^n$ is the topology defined by the open sets of the form $V(S)^C$ for some set of polynomials $S$.

## Remark

The variety $\mathbb{C}^n$ endowed with the Zariski topology is usually denoted by $\mathbb{A}^n$ or $\mathbb{A}^n_{\mathbb{C}}$.

# Some remarks

### Definition

Given any affine algebraic variety $Z \subseteq \mathbb{A}^n$, we endow it with the subspace topology coming from the Zariski topology of $\mathbb{A}^n$.

The Zariski topology is very different from the Euclidean topology.

- Every open set in the Zariski topology is also open in the Euclidean topology, but the converse is false.
- Open sets in the Zariski topology are very large. They are in fact dense (in both topologies). In particular, they are unbounded.
- Our intuition of product spaces needs to adapt: The closed subsets of $\mathbb{A}^1$ are finite collections of points, however those of $\mathbb{A}^2$ are not!

If we considered product top. in $\mathbb{A}^1 \times \mathbb{A}^1$: closed sets would be finite collections of pts.

# Morphisms of algebraic varieties

- In topology, one studies continuous maps;
- In differential geometry, one studies smooth maps;
- In complex geometry, one studies holomorphic maps;
- In algebraic geometry, varieties are defined by polynomials, so we study polynomial functions on them.

## Definition

A function $F : \mathbb{A}^n \to \mathbb{A}^m$ is a morphism if each component $F_i$ of $F$ is a polynomial in $n$ variables.

$$F(x) = (F_1(x), F_2(x), \ldots, F_m(x))$$

More generally, let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be affine varieties. A map $F : V \to W$ is a morphism of algebraic varieties if it is the restriction of a polynomial map $\tilde{F} : \mathbb{A}^n \to \mathbb{A}^m$.

## Examples

- A linear map $L : \mathbb{A}^n \to \mathbb{A}^n$ is an example of a morphism of affine varieties. It is an isomorphism if the matrix corresponding to $L$ is invertible.

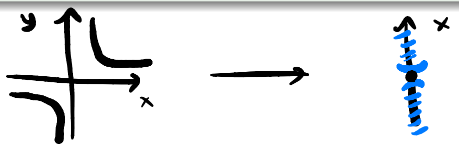
- Let $C$ $\subseteq \mathbb{A}^2$ be the plane parabola defined by the vanishing of $y - x^2$. The map

$$\mathbb{A}^1 \to C \subseteq \mathbb{A}^2$$

defined via $t \to (t, t^2)$ is an isomorphism. Its inverse is given by the (restriction of) the projection map

$$\mathbb{A}^2 \supset C \to \mathbb{A}^1, \quad (x, y) \mapsto x$$

- Morphisms don't send subvarieties to subvarieties. Consider the projection map $\pi : \mathbb{A}^2 \to \mathbb{A}^1$ to the first component. Note that the hyperbola $V(xy-1) = \{(t, t^{-1}) : t \neq 0\}$ gets mapped to $\mathbb{A}^1 \setminus \{0\}$, which is not Zariski-closed.



# Correspondence of ideals and affine varieties

# Ideal associated to a variety

Given a subvariety $V \subseteq \mathbb{A}^n$, we define its corresponding ideal $I(V)$ as follows:

$$I(V) = \{f \in \mathbb{C}[x_1, \dots, x_n] : f(P) = 0 \text{ for all } P \in V\}.$$

We have:

- If $V \subseteq W$, then $I(V) \supseteq I(W)$;
- $I(\emptyset) = \mathbb{C}[x_1, \dots, x_n]$ and $I(\mathbb{C}^n) = 0$;
- $I(\bigcup V_i) = \bigcap I(V_i)$.

# Hilbert's Nullstellensatz

## Nullstellensatz

For any ideal $\mathfrak{a} \subseteq \mathbb{C}[x_1, \ldots, x_n]$, we have

$$I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}},$$

where $\sqrt{\mathfrak{a}}$ is called the *radical* of the ideal.

## Definition

The radical of an ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is defined as follows:

$$\sqrt{I} = \{f \in \mathbb{C}[x_1, \ldots, x_n] : f^r \in I \text{ for some } r \geq 1\}.$$

Ex. $I = (x^2) \Rightarrow \sqrt{I} = (x)$.

$= \{f \cdot x^2 \mid f \in \mathbb{C}[x_1, \ldots, x_n]\}$

# Punchline of Nullstellensatz

The theorem tells us that there is a one-to-one correspondence between affine varieties and radical ideals of $\mathbb{C}[x_1, \ldots, x_n]$

$$\left\{ \begin{array}{c} \text{affine alg.} \\ \text{varieties} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{Radical ideals} \\ \text{of } \mathbb{C}[x_1, \ldots, x_n] \end{array} \right\}$$

## Warning

Just like the fundamental theorem of algebra, the Nullstellensatz fails if we work with real polynomials: You can check that $I = (x^2 + 1)$ and $J = (1)$ are both radical and correspond to the same variety.

# The coordinate ring of a variety

As we mentioned before, we study varieties through functions on them. In this setting, we study polynomial functions.

Let $V \subseteq \mathbb{A}^n$ be an affine variety. Given a polynomial in $n$ variables, it's restriction to $V$ defines a function $V \to \mathbb{C}$.

These functions define a $\mathbb{C}$-algebra which we call the **coordinate ring of $V$** and denote it by $\mathbb{C}[V]$.

Ring + $\mathbb{C}$- vector space.

**Example.** $\mathbb{C}[\mathbb{A}^n] = \mathbb{C}[x_1, \dots, x_n]$.

Note that there is a surjective ring homomorphism

$$\mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}[V],$$

with kernel precisely the ideal $I(V)$ of functions vanishing on $V$. In other words,

$$\mathbb{C}[V] \cong \mathbb{C}[x_1, \dots, x_n]/I(V).$$

(1st iso. thm.)

# The pullback map

A map $F : V \to W$ of affine algebraic varieties defines a map

$$F^{\#} : \mathbb{C}[W] \to \mathbb{C}[V]$$

via precomposition: $g \mapsto g \circ F$. $(g : W \to \mathbb{C}) \longmapsto (g \circ F : V \xrightarrow{F} W \xrightarrow{g} \mathbb{C})$

## Equivalence of categories

Every finitely generated reduced $\mathbb{C}$-algebra is isomorphic to the coordinate ring of some affine variety.

**Idea of proof.** Given $V \subseteq \mathbb{A}^n$, its associated $\mathbb{C}$-algebra is $\mathbb{C}[V]$. Conversely, any finitely generated reduce $\mathbb{C}$-algebra is of the form $\mathbb{C}[x_1, \ldots, x_n]/I$ for some fixed $n$ and ideal $I$. Its corresponding variety is $V(I)$.

f.g. reduced $\mathbb{C}$-alg $\quad \overset{\text{"1:1" up to iso.}}{\longleftrightarrow} \quad$ Varieties

* We can also talk about varieties without a notion of ambient space.

This requires the "Spec" construction.