A proof of Sobolev's Embedding Theorem for Compact Riemannian Manifolds

The source for most of the following is Chapter 2 of Thierry Aubin's, "Some Nonlinear Problems in Riemannian Geometry," 1998, Springer-Verlag. Page references in this document are to Aubin's text.

Let (M, g) be a smooth, n-dimensional Riemannian manifold. Define:

$$C^k(M) := \{k\text{-times differentiable real-valued functions from } M \text{ to } \mathbb{R}\}\,;$$

$$C^\infty(M) := \{\text{infinitely differentiable real-valued functions from } M \text{ to } \mathbb{R}\}\,;$$

$$\mathcal{F}_k^p := \left\{\varphi \in C^\infty(M) : |\nabla^l \varphi| \in L^p(M) \text{ for all } 0 \le l \le k\right\},$$

for real p, $1 \le p < \infty$, and nonnegative integer, k;

$$L^p(M) := \left\{\varphi : M \to \mathbb{R} : \varphi \text{ is measurable and } \int_M |\varphi|^p \, dV < \infty\right\},$$

where $dV = \sqrt{|g|}$, $\quad g = \det(g_{ij})$;

$$\mathcal{D}(M) := \{\varphi \in C^\infty(M) : \varphi \text{ has compact support}\}\,.$$

$\mathbb{C}_0^\infty(M)$ might be more natural notation than $\mathcal{D}(M)$. Also, Aubin uses a subscript on L^p spaces rather than a superscript, but I don't want to get into that habit, so I am using superscripts.

After digesting these definitions, finally we can define Sobolev spaces.

Definition. The *Sobolev space* $H_k^p(M)$ for p real, $1 \le p < \infty$ and k a nonnegative integer, is the completion of $\mathcal{F}_k^p$ with respect to the norm

$$\|\varphi\|_{H_k^p} := \sum_{l=0}^{k} \|\nabla^l \varphi\|_p.$$

Observe that $H_0^p(M) = L^p(M)$. Also, $\mathbb{H}_k := H_k^2$ is a Hilbert space under the L^2-inner product.

$\mathcal{F}_k^p$ contains only smooth functions. In general, a sequence in $\mathcal{F}_k^p$ will not converge in the H_k^p norm to a function in $\mathcal{F}_k^p$, so we need to complete the space to have anything useful. An alternate approach would have been to start with functions in L^p rather than completing the space of smooth functions in $\mathcal{F}_k^p$.

Definition. For $P, Q \in M$, let

$$d(P, Q) := \inf\{\text{lengths of curves from } P \text{ to } Q\}\,,$$

the length along the curve $c : I \to M$, where I is an interval in $\mathbb{R}$, being defined by the metric:

$$l(P, Q) := \int_P^Q \sqrt{\left\langle \frac{Dc}{dt}, \frac{Dc}{dt} \right\rangle}\, dt.$$

The distance function, $d(P, \cdot)^2$, needn't be smooth, but because of the triangle inequality, $d(P, \cdot)$ *is* Lipschitz (with constant 1). The function $d(P, \cdot)^2$ not being smooth complicates the proof of the following theorem:

Theorem 0.1 (Theorem 2.6 p. 34)**.** *For M a complete Riemannian manifold, $\dot{H}_k^p(M)$, the closure $\mathcal{D}(M)$ in $H_k^p(M)$, is equal to $H_k^p(M)$. That is, in a complete manifold, the smooth, compactly supported functions are dense in every Sobolev space.*

Proof. $\mathcal{F}_k^p$ is dense in $H_k^p(M)$, its completion, and so it suffices to show that $\mathcal{D}(M)$ is dense in $\mathcal{F}_k^p$. So let $\varphi \in \mathcal{F}_k^p$, and define

$$f(t) = \begin{cases} 1, & t < 0, \\ 1 - t, & 0 < t < 1, \\ 0, & t > 1. \end{cases}$$

Fix $P \in M$ and let

$$\varphi_j(Q) = \varphi(Q) f(d(P, Q) - j).$$

Then φ_j is compactly supported by the Hopf-Rinow Theorem, which states in part that any closed bounded subset of a complete manifold is compact. (This is our only use of completeness of the manifold, but it is absolutely critical.)

Now, φ_j need not be smooth, an issue we deal with below, but our immediate goal is to show that $\varphi_j \to \varphi$ in the $H_k^p(M)$ norm. Toward that end, observe that $|\varphi_j| = |\varphi|\,|f| \le |\varphi|$, and so we have

(1) $\varphi_j \in L^p(M)$,
(2) $\varphi_j \to \varphi$ pointwise everywhere, and
(3) $|\varphi_j| \le |\varphi|$ and $\varphi \in L^p(M)$.

By the Lebesgue Dominated Convergence Theorem (DCT) (the Riemannian metric puts a measure on the manifold, so DCT applies as it would on any measure space), $\varphi_j \to \varphi$ in $L^p(M)$. If $k = 0$, then we are done, since $H_0^p(M) = L^p(M)$.

Assume that $k \ge 1$. The function $d(P, \cdot)$ is Lipschitze, hence absolutely continuous, and hence its gradient exists *a. e.*. Hence, the following bound holds *a. e.*:

$$\begin{aligned}|\nabla\varphi_j| &= |f(d(P,\cdot)-j)\nabla\varphi + \varphi\nabla(f(d(P,\cdot)-j))| \\ &\le |\nabla\varphi| + |\varphi|\,|\nabla(f(d(P,\cdot)-j))| \\ &= |\nabla\varphi| + |\varphi|\,\left|f'(d(P,\cdot)-j)\right|\,|\nabla d(P,\cdot)| \\ &\le |\nabla\varphi| + |\varphi|\,.\end{aligned}$$

In the last inequality we used the fact that $|f'| \le 1$ *a. e.* and that all difference quotients appearing in the definition of $\nabla d(P,\cdot)$ have absolute value bounded by 1.

Then, since φ and $\nabla\varphi$ are in $L^p(M)$ (φ being in $H_k^p(M)$), $\nabla\varphi_j \in L^p(M)$. Thus,

(1) $\nabla\varphi_j \in L^p(M)$,
(2) $\nabla\varphi_j \to \nabla\varphi$ pointwise everywhere, and
(3) $|\nabla\varphi_j| \le |\nabla\varphi| + |\varphi|$ and $|\nabla\varphi| + |\varphi| \in L^p(M)$.

Applying the DCT as for φ_j, $\nabla\varphi_j \to \nabla\varphi$ in $L^p(M)$.

The higher derivatives are handled similarly. Thus, since all the gradients of φ_j up to the k-th order converge to φ in the $L^p(M)$ norm, φ_j converges to φ in $H_k^p(M)$.

To deal with the fact that φ_j needn't be smooth, we put a finite partition of unity (POU) on $\operatorname{supp}\varphi_j$, produce a sequence of smooth functions in each partition, each of whose gradients up to order k converge to φ_j in $L^p(M)$—hence insuring that the sequence converges to φ_j in H_k^p(the partition)—and form a sequence of functions on M from the POU that converges to φ_j in $H_k^p(M)$. Call this sequence of functions, $\{\alpha_{jk}\}_{j,k=1}^\infty$.

Construct a subsequence, $\{\beta_j\}_{j=1}^\infty$ of $\{\alpha_{jk}\}_{j,k=1}^\infty$ by choosing β_j from $\{\alpha_{jk}\}_{k=1}^\infty$ so that $\|\beta_j - \varphi_j\|_{H_k^p} < \|\varphi_j - \varphi\|_{H_k^p}$. Then $\{\beta_j\}_{j=1}^\infty$ is a sequence of functions in $\mathcal{D}(M)$ that converge to φ in $H_k^p(M)$—at least half as slowly as φ_j converges to φ—which completes the proof. □

Our next goal is to state and prove the Sobolev Embedding Theorem (SET) for compact manifolds, but first we need some definitions.

Definition. Let r be a nonnegative integer. Then

$$C_B^r := \left\{u \in C^r : |\nabla^l u| \text{ is bounded for all } 0 \le l \le r\right\},$$

$$\|u\|_{C_B^r} := \max_{0\le l\le r}\sup_{u\in M}|\nabla^l u|.$$

Let $\alpha \in (0,1)$. Then

$$C^\alpha := \left\{u \in C^1 : |u(P) - u(Q)| \le C d(P,Q)^\alpha\right\} \text{ for some constant } C,$$

$$C^{r+\alpha} := \left\{u \in C^r : \partial^\beta u \in C^\alpha \text{ for all } |\beta| \le r\right\}.$$

The $C^{r+\alpha}$ are called Hölder spaces. A norm for C^α is

$$\|u\|_{C^\alpha} := \sup|u| + \sup_{P\neq Q}\left\{|u(P)-u(Q)|\,d(P,Q)^{-\alpha}\right\}.$$

[Aubin does not define a norm for $C^{r+\alpha}$ in general, but a sum of the C^α norm for the function and its derivatives up to the r-th order is one possible norm.]

Theorem 0.2 (Theorem 2.20 p. 44, SET for compact manifolds)**.** *Let (M, g) be a compact Riemannian manifold of dimension n. There are two parts to the theorem.*

> ***First part:*** *Let $k, l \in \mathbb{Z}$, $k > l \geq 0$, $p, q \in \mathbb{R}$, $1 \leq q < p$ such that*
>
> $$\frac{1}{p} = \frac{1}{q} - \frac{k-l}{n}.$$
>
> *Then $H_k^q(M) \subseteq H_l^p(M)$, and the inclusion map $i : H_k^q(M) \hookrightarrow H_l^p(M)$ is continuous.*
>
> ***Second part:*** *If r is a nonnegative integer and*
>
> $$\frac{k-r}{n} > \frac{1}{q},$$
>
> *then $H_k^q \subseteq C_B^r$ and the inclusion map $i : H_k^q \hookrightarrow C_B^r$ is continuous. If, for $0 < \alpha < 1$, α real,*
>
> $$\frac{k-r-\alpha}{n} \geq \frac{1}{q},$$
>
> *then $H_k^q \subseteq C^{r+\alpha}$ and the inclusion map $i : H_k^q \hookrightarrow C^\alpha$ is continuous [Aubin claims nothing for the inclusion into $C^{r+\alpha}$.]*

The first part of the SET relates Sobolev spaces of different orders (the k and the l) and, most important, based on different Lebesgue spaces. I have never seen a treatment that relates Sobolev spaces from different L^p spaces: all I have seen of part one is the obvious statement that $k \geq l \geq 0 \Rightarrow H_k^p \subseteq H_l^p$. It is the second part that I am familiar with as the statement of the SET.

Comment: Aubin states on p. 36 that "We will mainly discuss the first part of the theorem [SET in its various forms], because the other part concerns local properties (except the continuity of the imbedding) and so there is no difference in the case of manifolds. One will find the complete proof in Theorem 2.21." Theorem 2.21 is SET for complete manifolds with bounded curvature and nonzero injectivity radius, which we will not have time for. Hence, we will never see a proof of the second part of this theorem.

We are going to give Aubin's own proof of the SET, which requires a proposition he proves before SET, but which we are going to prove after SET. This is:

Proposition 0.3 (Prop 2.11 p. 36)**.** *Let M be any smooth Riemannian manifold of dimension n. If $H_1^{q_0}(M) \hookrightarrow L^{p_0}(M)$ with*

$$\frac{1}{p_0} = \frac{1}{q_0} - \frac{1}{n}, \quad 1 \leq q_0 < n,$$

then $H_k^q(M) \hookrightarrow H_l^{p_l}(M)$ with

$$\frac{1}{p_l} = \frac{1}{q} - \frac{k-l}{n} > 0.$$

Proposition 2.11 says that once we are able to embed a Sobolev space of order 1 into a "complementary" Sobolev space of order 0 (a Lebesgue space), we automatically get embeddings of higher order Sobolev spaces into any other "complementary" Sobolev space of the approriate order. This will allow us to only deal with the gradient of functions in the proof of the SET, bootstrapping ourselves up to a Sobolev space of any order by this Proposition.

Proof. (Of ***first part*** *of the SET for compact manifolds—see Theorem 2.20 p. 44.)* To show that a space A is (continuously) embedded in a space B, we need only show that the B's norm is bounded by a constant times A's norm, for then any sequence converging in A will converge in B (which proves sequential continuity of the inclusion function). In our application, $C^\infty(M)$ will be dense in both the embedded and the embedding spaces (by Theorem 2.6 for Sobolev spaces and the fact that M is compact, so $\mathcal{D}(M) = C^\infty(M)$), so we need only establish the bound on the norms for smooth functions.

Since M is compact, we can obtain a finite open cover, $\{\Omega_i\}_{i=1}^N$ of M. [This is the critical use of compactness, which makes the proof "easy"]. Let $\{(\Omega_i, \varphi_i)\}$ be the corresponding charts, and let $\{\alpha_i\}$ be a smooth POU subordinate to $\{\Omega_i\}$. Assume we have shown that, for p, q such that $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$, that there exist constants $\{C_i\}_{i=1}^N$ such that for all $f \in C^\infty(M)$,

$$\|\alpha_i f\|_{L^p} \leq C_i \|\alpha_i\|_{H_1^q}. \tag{1}$$

If $\{f_k\}$ is a sequence in $\mathbb{C}^\infty(M)$ with $f_k \to 0$ in the H_1^q norm, then

$$\begin{aligned} \|f_k\|_{L^p} &= \|\sum_{i=1}^N \alpha_i f_k\|_{L^p} \leq \sum_{i=1}^N \|\alpha_i f_k\|_{L^p} \\ &\leq \sum_{i=1}^N C_i \|\alpha_i f_k\|_{H_1^q} \to 0, \end{aligned}$$

since each $\|\alpha_i f_k\|_{H_1^q} \to 0$. Notice how we have used the finiteness of the partition in a critical way. [Replace 0 by f and f_k by $f_k - f$ if one wishes to conclude that $f_k \to f$ in H_1^q and so in L^p.] Therefore, $H_1^q(M) \hookrightarrow L^p(M)$ and so by Proposition 2.11, $H_k^q(M) \hookrightarrow H_l^p(M)$ for any p, q such that $\frac{1}{p} = \frac{1}{q} - \frac{k-l}{n} > 0$, giving part 1 of the theorem.

It remains to establish the existence of the constants such that Equation (1) holds. This is ultimately a consequence of Sobolev's Lemma, via Corollary 0.7, which we state and prove below. To apply this corollary, observe that since α_i and f are both smooth,

$$\nabla(\alpha_i f) = \alpha_i \nabla f + f \nabla \alpha_i,$$

so, working in charts (and suppressing the homeomorphisms)

$$\begin{aligned}
\|\alpha_i f\|_{L^p} &\leq C(n,q)\, \|\nabla(\alpha_i f)\|_{L^q} \leq C(n,q)\left[\|\alpha_i \nabla f\|_{L^q} + \|f \nabla \alpha_i\|_{L^q}\right] \\
&\leq C(n,q)\left[\|\alpha_i\|_{L^\infty} \|\nabla f\|_{L^q} + \|\nabla \alpha_i\|_{L^\infty} \|f\|_{L^q}\right] \\
&\leq C(n,q) \max\left\{1, \|\nabla \alpha_i\|_{L^\infty}\right\}\left[\|f\|_{L^q} + \|\nabla f\|_{L^q}\right] \\
&\leq C_i \|f\|_{H_1^q},
\end{aligned}$$

where we used the fact that $\|\alpha_i\|_{L^\infty} \leq 1$, and where

$$C_i = C(n,q) \max\left\{1, \|\nabla \alpha_i\|_{L^\infty}\right\}.$$

Now, this result applies to the norms in the charts, but Aubin makes an argument that since the manifold is compact, the metric distorts the Euclidean norm on which the above calculations are based by no less than a minimum factor and no more than a maximum factor, which in the end does not affect our calculations (though it changes the constants, C_i). $\square$

We need to establish Corollary 0.7 to complete the first part of the proof. Corollary 0.7 is actually a corollary to a corollary of Sobolev's Lemma.

Lemma 0.4 (Sobolev's Lemma, Lemma 2.12 p. 37). *Let $p', q' > 1$ be real, and let $\lambda \in \mathbb{R}$ satisfy*

$$\frac{1}{p'} + \frac{1}{q'} + \frac{\lambda}{n} = 2.$$

If $0 < \lambda < n$, then there exists a constant $K(p', q', n)$ such that for all $f \in L^{q'}(\mathbb{R}^n)$, $g \in L^{p'}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x) g(y)}{\|x-y\|^\lambda}\, dx\, dy \leq K(p', q', n)\, \|f\|_{L^{q'}} \|g\|_{L^{p'}},$$

$\|x - y\|$ being the Euclidean norm.

Aubin does not give a proof of Sobolev's Lemma, but I believe the applied notes contain one, and we should at least look at its proof.

Corollary 0.5 (Corollary 2.12 p. 37)**.** *Let $\lambda \in \mathbb{R}$, $0 < \lambda < n$, $q' \in \mathbb{R}$, $q' > 1$. If $r \in \mathbb{R}$ is such that $r > 1$ and*

$$\frac{1}{r} = \frac{\lambda}{n} + \frac{1}{q'} - 1,$$

then

$$h(y) := \int_{R^n} \frac{f(x)}{\|x-y\|^\lambda} \, dx \in L^r(\mathbb{R}^n) \text{ when } f \in L^{q'}(\mathbb{R}^n),$$

and there exists a constant $C(\lambda, q', n)$ such that

$$\|h\|_{L^r} \leq C(\lambda, q', n) \|f\|_{L^{q'}}.$$

Proof. By Sobolev's lemma, for all $g \in L^{p'}(\mathbb{R}^n)$ with $\frac{1}{r} + \frac{1}{p'} = 1$,

$$\begin{aligned}\int_{\mathbb{R}^n} h(y)g(y)\,dy &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{\|x-y\|^\lambda}\,dx\,dy \\ &\leq K(p', q', n) \|f\|_{L^{q'}} \|g\|_{L^{p'}}.\end{aligned}$$

Therefore, $h \in (L^{p'})^* \cong L^r$ [that is, h can be identified with a continuous linear functional on $L^{p'}$ by the Riesz representation theorem]. Also, $\|h\|_{L^r} \leq K(p', q', n) \|f\|_{L^{q'}}$, which we can see by viewing it as a linear functional. □

Corollary 0.6. *[Unnamed lemma on the bottom of p. 37] For p, q real, $p > 1$, $1 < q < n$, and such that $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$, there exists a constant $C(n, q)$ such that*

$$\|\varphi\|_{L^p} \leq C(n, q) \|\nabla\varphi\|_{L^q}.$$

Proof. Fix $x \in \mathbb{R}^n$ and put spherical coordinates (r, θ) centered at x [θ represents $n-1$ variables specifying the point of intersection with S^{n-1} of the ray through the origin and the point x]. Since φ is smooth and compactly supported,

$$\begin{aligned}\varphi(x) = -\int_0^\infty \frac{\partial\varphi(r,\theta)}{\partial r}\,dr &= -\int_0^\infty \|x-y\|^{1-n} \frac{\partial\varphi(r,\theta)}{\partial r} r^{n-1}\,dr \\ \Rightarrow |\varphi(x)| &\leq \int_0^\infty \|x-y\|^{1-n} |\nabla\varphi(r,\theta)|\, r^{n-1}\,dr,\end{aligned}$$

since $|\nabla\varphi(r,\theta)| \leq \frac{\partial\varphi(r,\theta)}{\partial r}$. Integrating over the θ variables gives

$$\begin{aligned}\omega_{n-1} |\varphi(x)| &\leq \int_{\mathbb{R}^n} \frac{|\nabla\varphi(y)|}{\|x-y\|^{n-1}}\,dy \\ &\Rightarrow \|\varphi\|_{L^p} \leq C(n, q) \|\nabla\varphi\|_{L^q}\end{aligned}$$

by applying Corollary 2.12 with $f = |\nabla\varphi(y)|$, $\lambda = n-1$, $r = p$, and $q' = q$—so

$$\frac{1}{r} = \frac{\lambda}{n} + \frac{1}{q'} - 1,$$

since

$$\frac{1}{p} = \frac{n-1}{n} + \frac{1}{q} - 1 = 1 - \frac{1}{n} + \frac{1}{q} - 1 = \frac{1}{q} - \frac{1}{n}.$$

□

Comment: Instead of using Corollary 0.7 in the proof of the SET for compact manifolds, Aubin uses the more powerful result in Theorem 2.14, which we state below. What makes Theorem 2.14 stronger is that the constant stated is the best possible, but that is something we do not need.

Theorem 0.7 (Theorem 2.14 p. 39)**.** *If* $1 \le q < n$ *and*

$$\frac{1}{p} = \frac{1}{q} - \frac{1}{n},$$

then for all $\varphi \in H_1^q(\mathbb{R}^n)$,

$$\|\varphi\|_{L^p} \le \mathbb{K}(n,q)\, \|\nabla\varphi\|_{L^q},$$

where

$$\mathbb{K}(n,q) = \begin{cases} \frac{q-1}{n-q}\left[\frac{n-q}{n(q-1)}\right]^{1/q}\left[\frac{\Gamma(n+1)}{\Gamma(\frac{n}{q})\Gamma(n+1-\frac{n}{q})\omega_{n-1}}\right]^{1/n}, & 1 < q < n, \\ \frac{1}{n}\left[\frac{n}{\omega_{n-1}}\right]^{1/n}. & q = 1, \end{cases}$$

Here, ω_{n-1} *is the volume of the unit sphere in* $\mathbb{R}^n$ *(that is, of* S^{n-1}*).*

$\mathbb{K}(n,q)$ *is the norm of the embedding* $H_1^q \hookrightarrow L^p$ *and is attained by the functions*

$$\varphi(x) = (\lambda + \|x\|^{q/(q-1)})^{1-n/q},$$

where λ *is any positive real number.*