CAUCHY, POMPEIU, GREEN, AND BIOT-SAVART

JAMES P. KELLIHER[2]

Abstract. The main purpose of this paper is pedagogical. We show how the divergence theorem can be used to prove a generalization of Cauchy's integral theorem that applies to a continuous complex-valued function, whether differentiable or not. We use this generalization to obtain the Cauchy-Pompeiu integral formula, a generalization of Cauchy's integral formula for the value of a function at a point. We then demonstrate the connections among the Cauchy-Pompeiu integral formula, the Biot-Savart law, and Green's representation formula in the context of the study of 2D incompressible fluids. We close by obtaining a generalization of the Cauchy integral formula for the derivative of a function.

Compiled on Thursday 19 October 2023 at 11:03

Contents

1. Path integrals and the divergence theorem

We begin by recalling the definition of contour integrals, real and complex:

Definition 1.1. *Let $C \subseteq \mathbb{R}^2$ be a curve parameterized by a path $\gamma\colon [a,b] \to C$ that is Lipschitz continuous, and assume that $f\colon C \to \mathbb{R}$ is continuous. We define the real contour integral by*

$$\int\!\!\!\!\!\!\mathbb{R}_{\gamma} f = \int\!\!\!\!\!\!\mathbb{R}_{C} f := \int_a^b f(\gamma(t))|\gamma'(t)|\,dt.$$

If $\mathbf{u}\colon C \to \mathbb{R}$ is continuous with $\mathbf{u} = (u^1, u^2)$, then $\int\!\!\!\!\!\mathbb{R}_C \mathbf{u}$ is vector-valued with

$$\int\!\!\!\!\!\!\mathbb{R}_{\gamma} u^j = \int\!\!\!\!\!\!\mathbb{R}_{C} u^j := \int_a^b u^j(\gamma(t))|\gamma'(t)|\,dt, \quad j = 1, 2.$$

Definition 1.2. *Let $C \subseteq \mathbb{C}$ be a curve parameterized by a path $\gamma\colon [a,b] \to C$ that is Lipschitz continuous, and assume that $f\colon C \to \mathbb{C}$ is continuous. We define the complex contour integral by*

$$\int_\gamma f := \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

We will sometimes also write

$$\int\!\!\!\!\!\!\mathbb{C}_{C} f := \int_\gamma f$$

when we don't wish to introduce a specific path, though this only defines the integral up to sign.

By assuming in Definitions 1.1 and 1.2 that the paths are Lipschitz continuous, we are assured that $\gamma'(t)$ is bounded and defined almost everywhere along the curve. We will always assume Lipschitz continuity of all paths without explicitly stating so.

Lemma 1.3 is useful for switching between viewing functions as defined on $\mathbb{R}^2$ and on $\mathbb{C}$.

Lemma 1.3. *Define a map,* $\vec{\cdot}\colon \mathbb{C} \to \mathbb{R}^2$, *by* $\overrightarrow{x+iy} = (x, y)$. *Let* $z, w \in \mathbb{C}$ *and* $\cdot$ *be the usual dot (inner) product of Euclidean vectors. Then*

$$\begin{aligned} \operatorname{Re}(zw) &= \vec{\overline{z}} \cdot \vec{w}, \\ \operatorname{Im}(zw) &= -\vec{\overline{z}} \cdot \vec{w}^{\perp}. \end{aligned} \tag{1.1}$$

(Here, and in what follows, if $\mathbf{x} = (x, y)$ *is a vector then*

$$\mathbf{x}^{\perp} := (-y, x).$$

Hence, $\mathbf{x}^{\perp}$ *is* $\mathbf{x}$ *rotated 90 degrees counterclockwise.)*

Also, f *is analytic in some domain* Ω *if and only if* $\operatorname{div} \vec{\overline{f}} = \operatorname{curl} \vec{\overline{f}} = 0$ *in* Ω, *where*

$$\operatorname{div} \mathbf{u} := \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}, \quad \operatorname{curl} \mathbf{u} := \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$$

are the divergence and (scalar) curl of $\mathbf{u}$.

Proof. Letting $z = x + iy$, $w = u + iv$, we have

$$zw = xu - vy + i(xv + yu) = (x, -y) \cdot (u, v) - i(x, -y)(-v, u) = \vec{\overline{z}} \cdot \vec{w} - i\vec{\overline{z}} \cdot \vec{w}^{\perp}.$$

That f analytic gives $\operatorname{div} \vec{\overline{f}} = \operatorname{curl} \vec{\overline{f}} = 0$ follows from the Cauchy-Riemann equations. □

We mentioned that analytic (by which we always mean complex-analytic) functions satisfy the Cauchy-Riemann equations. Another way of writing this is to define the *Wirtinger operators*,

$$\partial_z := \frac{\partial_x - i\partial_y}{2}, \quad \partial_{\overline{z}} := \frac{\partial_x + i\partial_y}{2}. \tag{1.2}$$

Then f satisfies the Cauchy-Riemann equations if and only if $\partial_{\overline{z}} f = 0$.

The Wirtinger operators have great utility because ∂_z and $\partial_{\overline{z}}$ operate independently; that is, abusing notation somewhat,

$$\partial_z f(\overline{z}) = \partial_{\overline{z}} f(z) = 0. \tag{1.3}$$

With Lemma 1.3, we can give the relationship between real and complex contour integrals.

Lemma 1.4. *Let* $\gamma\colon [a, b] \to \mathbb{C}$ *be a path on which the complex-valued function* f *is continuous. Let* $\boldsymbol{\tau}$ *be the unit tangent vector in the direction of* $\vec{\gamma}$ *and* $\boldsymbol{n}$ *the associated unit normal, with* $(\boldsymbol{n}, \boldsymbol{\tau})$ *in the standard orientation of* $(\mathbf{e}_1, \mathbf{e}_2)$. *Let* $C = \operatorname{image} \gamma$. *Then*

$$\int_\gamma f = \mathbb{R}\int_C \vec{\overline{f}} \cdot \boldsymbol{\tau} + i\mathbb{R}\int_C \vec{\overline{f}} \cdot \boldsymbol{n}.$$

Proof. We have,

$$\operatorname{Re} \int_\gamma f = \int_a^b \operatorname{Re}\left[f(\gamma(t))\gamma'(t)\right] \, dt = \int_a^b \vec{\overline{f}}(\gamma(t)) \cdot \overrightarrow{\gamma'(t)} \, dt.$$

Here, we applied $(1.1)_1$ with $z = f(\gamma(t))$, $w = \gamma'(t)$. Now, we are free to reparameterize γ as we see fit, so let us just assume up front that γ is parameterized by arc length, meaning that $|\gamma'(t)| = |\vec{\gamma}'(t)| = 1$ for all t. It follows, then, that

$$\overrightarrow{\gamma'(t)} = \vec{\gamma}'(t) = \boldsymbol{\tau},$$

the unit tangent vector in the direction that $\gamma(t)$ moves. Then we can see that

$$\boldsymbol{\tau}^\perp = -\boldsymbol{n}.$$

Hence,

$$\operatorname{Re} \int_\gamma f = \int_a^b \vec{\bar{f}}(\gamma(t)) \cdot \overrightarrow{\gamma'(t)}\, dt = \int_a^b \vec{\bar{f}}(\gamma(t)) \cdot \overrightarrow{\gamma'(t)} |\gamma'(t)|\, dt = \oint_C \vec{\bar{f}} \cdot \boldsymbol{\tau}.$$

Similarly,

$$\operatorname{Im} \int_\gamma f = \int_a^b \operatorname{Im} \left[f(\gamma(t))\gamma'(t) \right]\, dt = -\int_a^b \vec{\bar{f}}(\gamma(t)) \cdot \overrightarrow{\gamma'(t)}^\perp\, dt.$$

Here, we applied $(1.1)_2$ with $z = f(\gamma(t))$, $w = \gamma'(t)$. But from above, we know that

$$-\overrightarrow{\gamma'(t)}^\perp = -\boldsymbol{\tau}^\perp = \boldsymbol{n},$$

so

$$\operatorname{Im} \int_\gamma f = \oint_C \vec{\bar{f}} \cdot \boldsymbol{n}. \qquad \square$$

In all that follows, we will assume that $\Omega \subseteq \mathbb{R}^2$ is a domain with compact boundary $\Gamma := \partial\Omega$ having at least Lipschitz regularity and having a finite number of boundary components, $\Gamma_1, \dots, \Gamma_{N+1}$. Because its boundary is compact, Ω is either a bounded domain or the exterior of one or more bounded domains. If Ω is bounded, we assume that Γ_{N+1} bounds the unbounded component of Ω or Ω^C.

On each Γ_j we will define unit normal and tangent vectors, $(\boldsymbol{n}, \boldsymbol{\tau})$, in the same orientation as the standard unit vectors $(\mathbf{e}_1, \mathbf{e}_2)$ and where we assume that $\boldsymbol{n}$ is outward-directed. Equivalently, as one moves in the direction of $\boldsymbol{\tau}$ along Γ_j, Ω is to the left and $\boldsymbol{n}$ points to the right.

We will use the divergence theorem in the following form:

Theorem 1.5. *Suppose that* $\mathbf{u}$ *is continuously differentiable on a neighborhood of* $\overline{\Omega}$. *Then*

$$\oint_{\partial\Omega} \mathbf{u} \cdot \boldsymbol{n} = \int_\Omega \operatorname{div} \mathbf{u}.$$

In Theorem 1.5, the boundary integral is a sum of $N+1$ integrals over each boundary component, the relative orientation of those boundary components being very important.

We will also frequently use the following corollary to the divergence theorem, which we will simply refer to as "integration by parts":

Corollary 1.6 ("Integration by parts")**.** *Assume that the vector-valued function* $\boldsymbol{X}$ *and the scalar-valued function* f *are continuously differentiable on a neighborhood of* $\overline{\Omega}$. *Then*

$$\int_\Omega \operatorname{div} \boldsymbol{X}\, f + \int_\Omega \boldsymbol{X} \cdot \nabla f = \int_{\partial\Omega} (\boldsymbol{X} \cdot \boldsymbol{n})\, f.$$

Proof. Apply Theorem 1.5 with $\mathbf{u} = f\boldsymbol{X}$ and use that

$$\operatorname{div} \mathbf{u} = f \operatorname{div} \boldsymbol{X} + \nabla f \cdot \boldsymbol{X}. \qquad \square$$

Finally, we note that both Theorem 1.5 and Corollary 1.6 require less regularity of $\mathbf{u}$ than we have assumed, but we will not investigate such issues in this paper.

2. A generalization of Cauchy's integral theorem

We will use the divergence theorem to prove Theorem 2.1, a generalization of Cauchy's integral theorem.

Theorem 2.1. *Let* $f \colon \Omega \to \mathbb{C}$ *with* $\vec{f}$ *continuously differentiable in a neighborhood of* $\overline{\Omega}$*. Then*

$$\int_\gamma f = 2i \int_\Omega \partial_{\overline{z}} f = \int_\Omega (\operatorname{curl} \vec{\overline{f}} + i \operatorname{div} \vec{\overline{f}}).$$

Here, the integral over $\gamma = \partial\Omega$ *is a complex contour integral and the integrals over* Ω *are the usual area integral (of the real and imaginary parts). Also,* $\partial_{\overline{z}}$ *is defined in (1.2). Moreover, if* f *is (complex) analytic on a neighborhood of* $\overline{\Omega}$ *then*

$$\int_\gamma f = 0.$$

Remark 2.2. *Assuming that* $\vec{f}$ *is continuously differentiable does not mean that* f *is complex-differentiable; so the first partial derivatives of* f *are continuous, but need not satisfy the Cauchy-Riemann equations.*

Proof. Let γ_j parameterize Γ_j, $j = 1, \dots, N+1$, oriented as described above. Thus, applying Lemma 1.4, Theorem 1.5, and Lemma 2.3,

$$\begin{aligned}
\int_\gamma f &= \sum_j \int_{\gamma_j} f = \sum_j \oint_{C_j} \vec{\overline{f}} \cdot \boldsymbol{\tau} + i \sum_j \oint_{C_j} \vec{\overline{f}} \cdot \boldsymbol{n} = \oint_{\partial\Omega} \vec{\overline{f}} \cdot \boldsymbol{\tau} + i \oint_{\partial\Omega} \vec{\overline{f}} \cdot \boldsymbol{n} \\
&= -\oint_{\partial\Omega} \vec{\overline{f}}^{\perp} \cdot \boldsymbol{n} + i \oint_{\partial\Omega} \vec{\overline{f}} \cdot \boldsymbol{n} = -\int_\Omega \operatorname{div} \vec{\overline{f}}^{\perp} + i \int_\Omega \operatorname{div} \vec{\overline{f}} \\
&= \int_\Omega (\operatorname{curl} \vec{\overline{f}} + i \operatorname{div} \vec{\overline{f}}) = \int_\Omega 2i \partial_{\overline{z}} f.
\end{aligned}$$

We used that $\boldsymbol{\tau}^\perp = -\boldsymbol{n}$ and that $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^\perp \cdot \mathbf{y}^\perp$ for any vectors $\mathbf{x}$, $\mathbf{y}$. We also used that for any vector field $\mathbf{v} = (v^1, v^2)$,

$$-\operatorname{div} \mathbf{v}^\perp = -\operatorname{div}(-v^2, v^1) = -\left(\partial_1(-v^2) + \partial_2 v^1\right) = \partial_1 v^2 - \partial_2 v^1 = \operatorname{curl} \mathbf{v}$$

giving $-\operatorname{div} \vec{\overline{f}}^{\perp} = \operatorname{curl} \overline{f}$.

Finally, $\operatorname{div} \vec{\overline{f}}$ and $\operatorname{curl} \vec{\overline{f}}$ both vanish by Lemma 1.3 when f is analytic (equivalently, $\partial_{\overline{z}} f = 0$), completing the proof. □

Lemma 2.3 lists the many relations among the divergence, curl, ∂_z, and $\partial_{\overline{z}}$ operators, one of which we used above in the proof of Theorem 2.1.

Lemma 2.3. *For* $\vec{f}$ *differentiable,*

$$\begin{aligned}
\operatorname{div} \vec{f} + i \operatorname{curl} \vec{f} &= 2\partial_z f, & \operatorname{div} \vec{\overline{f}} + i \operatorname{curl} \vec{\overline{f}} &= 2\partial_z \overline{f}, \\
\operatorname{div} \vec{\overline{f}} - i \operatorname{curl} \vec{\overline{f}} &= 2\partial_{\overline{z}} f, & \operatorname{div} \vec{f} - i \operatorname{curl} \vec{f} &= 2\partial_{\overline{z}} \overline{f},
\end{aligned}$$

$$\begin{aligned}
\operatorname{curl} \vec{f} - i \operatorname{div} \vec{f} &= -2i\partial_z f, & \operatorname{curl} \vec{\overline{f}} - i \operatorname{div} \vec{\overline{f}} &= -2i\partial_z \overline{f}, \\
\operatorname{curl} \vec{\overline{f}} + i \operatorname{div} \vec{\overline{f}} &= 2i\partial_{\overline{z}} f, & \operatorname{curl} \vec{f} + i \operatorname{div} \vec{f} &= 2i\partial_{\overline{z}} \overline{f},
\end{aligned}$$

and

$$\operatorname{div} \vec{f} = \partial_z f + \partial_{\overline{z}} \overline{f}, \quad \operatorname{curl} \vec{f} = -i[\partial_z f - \partial_{\overline{z}} \overline{f}],$$
$$\operatorname{div} \vec{\overline{f}} = \partial_z \overline{f} + \partial_{\overline{z}} f, \quad \operatorname{curl} \vec{\overline{f}} = -i[\partial_z \overline{f} - \partial_{\overline{z}} f].$$

Proof. First we compute that

$$\operatorname{div} \vec{\overline{f}} = \operatorname{div}(u, -v) = \partial_x u - \partial_y v, \quad \operatorname{curl} \vec{\overline{f}} = \operatorname{curl}(u, -v) = -(\partial_x v + \partial_y u).$$

For the first set of identities, we calculate,

$$2\partial_z f = (\partial_x - i\partial_y)(u + iv) = \partial_x u + \partial_y v + i(\partial_x v - \partial_y u) = \operatorname{div} \vec{f} + i \operatorname{curl} \vec{f},$$
$$2\partial_z \overline{f} = (\partial_x - i\partial_y)(u - iv) = \partial_x u - \partial_y v - i(\partial_x v + \partial_y u) = \operatorname{div} \vec{\overline{f}} + i \operatorname{curl} \vec{\overline{f}},$$
$$2\partial_{\overline{z}} f = (\partial_x + i\partial_y)(u + iv) = \partial_x u - \partial_y v + i(\partial_x v + \partial_y u) = \operatorname{div} \vec{\overline{f}} - i \operatorname{curl} \vec{\overline{f}},$$
$$2\partial_{\overline{z}} \overline{f} = (\partial_x + i\partial_y)(u - iv) = \partial_x u + \partial_y v - i(\partial_x v - \partial_y u) = \operatorname{div} \vec{f} - i \operatorname{curl} \vec{f}.$$

The second and third set of identities then follow from the first set. □

3. A generalization of Cauchy's integral formula: Pompeiu

We can also generalize Cauchy's integral formula for the value of a function, as in Theorem 3.1. The idea of such a generalization goes back at least to Pompeiu [4], and led to the Cauchy-Pompeiu integral formula.

Theorem 3.1 (Cauchy-Pompeiu integral formula)**.** *Make the same assumptions as in Theorem 2.1. Then for any $a \in \Omega$,*

$$\begin{aligned} f(a) &= \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - a}\, dz - \frac{1}{\pi} \int_\Omega \frac{\partial_{\overline{z}} f(z)}{z - a}\, d\vec{z} \\ &= \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - a}\, dz - \frac{1}{2\pi i} \int_\Omega \frac{\operatorname{curl} \vec{\overline{f}}(\vec{z}) + i \operatorname{div} \vec{\overline{f}}(\vec{z})}{z - a}\, d\vec{z}. \end{aligned}$$

In Theorem 3.1, $\int_\Omega$ is the area integral of the real and imaginary parts of the integrand. The first expression for $f(a)$ is how the Cauchy-Pompeiu formula is usually written in terms of the $\partial_{\overline{z}}$ operator, though not using the language of differential forms. The second expression for $f(a)$ is in a form more directly applicable to incompressible fluid mechanics, as we will see in Section 5.

Proof. We will apply Theorem 2.1 to $f(z)/(z - a)$ and make an argument to cut out the singularity that results. To do this, define

$$\Omega_\varepsilon := \Omega \setminus B_\varepsilon(a)$$

for all $\varepsilon > 0$ sufficiently small that $\overline{B}_\varepsilon(a) \subseteq \Omega$. Then we can apply Theorem 2.1 to give

$$\oint_{\partial\Omega_\varepsilon} \frac{f(z)}{z - a}\, dz = 2i \int_{\Omega_\varepsilon} \partial_{\overline{z}} \frac{f(z)}{z - a}\, d\vec{z} = 2i \int_{\Omega_\varepsilon} \frac{\partial_{\overline{z}} f(z)}{z - a}\, d\vec{z}$$

by virtue of (1.3). This holds for all $\varepsilon > 0$ and so holds after taking $\varepsilon \to 0$, giving

$$\begin{aligned} \oint_{\partial\Omega} \frac{f(z)}{z - a}\, dz &= \lim_{\varepsilon \to 0} \oint_{\partial\Omega_\varepsilon} \frac{f(z)}{z - a}\, dz + \lim_{\varepsilon \to 0} \oint_{\partial B_\varepsilon(a)} \frac{f(z)}{z - a}\, dz \\ &= 2i \int_\Omega \frac{\partial_{\overline{z}} f(z)}{z - a}\, d\vec{z} + \lim_{\varepsilon \to 0} \oint_{\partial B_\varepsilon(a)} \frac{f(z)}{z - a}\, dz. \end{aligned}$$

But f is continuous, so

$$\lim_{\varepsilon\to 0}\oint_{\partial B_\varepsilon(a)} \frac{f(z)}{z-a}\,dz = \lim_{\varepsilon\to 0}\int_0^{2\pi} \frac{f(a+\varepsilon e^{it})}{e^{it}} i e^{it}\,dt = 2\pi i f(a),$$

from which, along with Lemma 2.3, the result follows. $\square$

4. Green's Representation Formula

Here and in the following section we discuss some connections between our generalizations of Cauchy's theorems and 2D incompressible fluid mechanics.

The velocity of an incompressible fluid is represented by a divergence-free vector field $\mathbf{u}$ on Ω. In most incompressible fluid applications, such as when working with the Euler equations, $\mathbf{u}\cdot\boldsymbol{n} = 0$ on the boundary, meaning that no fluid leaves or enters the container, Ω. It is natural, then, to work in the function space,

$$H := \{\mathbf{u} \in (L^2(\Omega))^2 \colon \operatorname{div}\mathbf{u} = 0,\ \mathbf{u}\cdot\boldsymbol{n} = 0\}, \tag{4.1}$$

endowed with the $L^2(\Omega)$ norm.

In this definition of H, the divergence is defined in terms of distributions (or as an element of $(H^{-1}(\Omega))$ and the normal component in terms of a trace. These technicalities will not, however, much concern us, for we will be working with vector fields whose gradient also lies in $L^2(\Omega)$; that is, with vector fields in $H \cap H^1(\Omega)^2$.

We will need to impose some additional constraints on the domain Ω, however. We assume that $\Omega \subseteq \mathbb{C}$ is a domain with compact boundary $\Gamma := \partial\Omega$ having at least Lipschitz regularity and having a finite number of boundary components, $\Gamma_1, \dots, \Gamma_{N+1}$. Because its boundary is compact, Ω is either a bounded domain or the exterior of a bounded domain; assume that Γ_{N+1} bounds the unbounded component of Ω or Ω^C.

Define H as in (4.1) and define the subspace of *harmonic vector fields*,

$$H_c = \{\mathbf{u} \in H \colon \operatorname{curl}\mathbf{u} = 0\}.$$

As a part of the Hodge decomposition of L^2 vector fields (a subject for another time), we have

$$H = H_0 \oplus H_c,$$

where the direct sum meaning that any $\mathbf{u} \in H$ can be written uniquely as a sum of vector field in H_0 and a vector field in H_c. Moreover, the direct sum is orthogonal in H, meaning that $(\mathbf{v}, \mathbf{w}) = 0$ for any $\mathbf{v} \in H_0$, $\mathbf{w} \in H_c$.

So let $\mathbf{u} \in H \cap H^1(\Omega)^2$. Fix x_0 on one boundary component, say Γ_{N+1}. For $x \in \Omega$, let γ be any path from x_0 to x lying in Ω. We can do this, since we have assumed that Ω is connected. Define

$$\psi(x) = -\int_\gamma \mathbf{u}^\perp \cdot \boldsymbol{\tau},$$

where here $\boldsymbol{\tau}$ is the unit vector in the direction of γ. Then the divergence-free condition on $\mathbf{u}$ together with $\mathbf{u}\cdot\boldsymbol{n} = 0$ on each boundary component allow us to apply the divergence theorem to conclude that ψ is independent of the chosen path. By choosing γ so that near x it is either horizontal or vertical, one can show that $\nabla\psi = -\mathbf{u}^\perp$, and hence that $\mathbf{u} = \nabla^\perp\psi$.

This is, in fact, a classical construction: noting that $\operatorname{curl}(-\mathbf{u}^\perp) = \operatorname{div}\mathbf{u} = 0$, this is the same manner in which one can obtain the potential for a curl-free vector field. In general, such a potential exists only for simply connected domains, which gives the independence of

ψ on the choice of path, but $\mathbf{u} \cdot \boldsymbol{n} = 0$ provides the independence in our setting for multiply connected domains. It also shows that ψ is constant on each boundary component.

Let $\omega := \operatorname{curl} \mathbf{u}$. Since $\Delta\psi = \operatorname{div} \nabla\psi = -\operatorname{div} \mathbf{u}^\perp = \operatorname{curl} \mathbf{u} = \omega$, ψ satisfies

$$\begin{cases} \Delta\psi = \omega & \text{in } \Omega, \\ \psi = C_j & \text{on } \Gamma_j \end{cases} \tag{4.2}$$

for some constants C_j.

Now restrict $\mathbf{u}$ to lie in $H_0 \cap H^1(\Omega)^2$, meaning it has no harmonic component. Since there is only one boundary component, we can choose the one constant $C_1 = 0$, since adding a constant to ψ does not change $\mathbf{u} = \nabla^\perp\psi$. Then (4.2) becomes

$$\begin{cases} \Delta\psi = \omega & \text{in } \Omega, \\ \psi = 0 & \text{on } \Gamma_j. \end{cases} \tag{4.3}$$

Now, Green's representation formula for the solution to (4.3) can be written in the form,

$$\begin{aligned} \psi(x) &= -\int_{\partial\Omega} \Phi(x-y)\, \boldsymbol{n} \cdot \nabla\psi(y)\, dy + \int_{\partial\Omega} (\nabla\Phi \cdot \boldsymbol{n})(x-y)\psi(y)\, dy + \int_{\Omega} \Phi(x-y)\Delta(y)\, dy \\ &= -\int_{\partial\Omega} \Phi(x-y)\, (\mathbf{u} \cdot \boldsymbol{\tau})(y)\, dy + \int_{\Omega} \Phi(x-y)\omega(y)\, dy, \end{aligned}$$

where

$$\Phi(x) := (2\pi)^{-1} \log|x|$$

is the fundamental solution to the Laplacian. (See, for instance, [2] Section 2.2.4(a) Equation (25)[1].) We also used that $\boldsymbol{n} \cdot \nabla\psi = \boldsymbol{n}^\perp \cdot \nabla^\perp\psi = \mathbf{u} \cdot \boldsymbol{\tau}$.

Defining the *Biot-Savart* kernel,

$$K(x) := \nabla^\perp\Phi(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}, \tag{4.4}$$

we can take the gradient of our expression for $\psi(x)$, giving

$$\mathbf{u}(x) = \nabla^\perp\psi(x) = -\int_{\partial\Omega} K(x-y)\, (\mathbf{u} \cdot \boldsymbol{\tau})(y)\, dy + \int_{\Omega} K(x-y)\omega(y)\, dy.$$

What we have proved is the following:

Theorem 4.1. *For any $\mathbf{u} \in H_0 \cap H^1(\Omega)^2$,*

$$\mathbf{u}(x) = \nabla^\perp\psi(x) = -\int_{\partial\Omega} K(x-y)\, (\mathbf{u} \cdot \boldsymbol{\tau})(y)\, dy + \int_{\Omega} K(x-y)\omega(y)\, dy.$$

Let $\mathbf{u}_\Gamma$, $\mathbf{u}_\Omega$ be the boundary, area integral in Theorem 4.1. Then in Ω, $\operatorname{div} \mathbf{u}_\Gamma = \operatorname{curl} \mathbf{u}_\Gamma = 0$, because $\operatorname{div}_x K(x-y) = \operatorname{curl}_x K(x-y) = 0$ for y away from x. This means that $\overleftarrow{\mathbf{u}_\Gamma}$ is analytic in D. If $\omega = \operatorname{curl} \mathbf{u} \in L^\infty(\Omega)$, say, then, although this is not immediately obvious, $\mathbf{u}$, $\mathbf{u}_\Gamma$, and $\mathbf{u}_\Omega$ are all continuous on $\overline{\Omega}$, so $\overleftarrow{\mathbf{u}_\Gamma}$ is analytic on D and continuous up to the boundary.

We cannot use Theorem 4.1 directly to recover a velocity field $\mathbf{u}$ from its curl ω, unless we know the value of the velocity on the boundary, though we can still use it effectively if we have some independent bounds on the value of $\mathbf{u}$ on the boundary. The boundary integral in Green's representation theorem, and so, ultimately, the boundary integral in Theorem 4.1, can be replaced by modifying the kernel in the area integral—this leads to Green's functions.

[1]Evans uses the opposite sign convention for the fundamental solution to the Laplacian, but is solving $\Delta\psi = -\omega$. So the representation formula here has the opposite sign to that in [2]

Although we will not follow the story that leads to a Green's function for Ω, we note that using it, one easily obtains a modified Biot-Savart kernel, K_Ω, which eliminates the boundary term in Theorem 4.1. One then has

$$\mathbf{u}(x) = \int_\Omega K_\Omega(x, y)\omega(y)\, dy. \tag{4.5}$$

The kernel K_Ω has the great advantage over $K(x - y)$ that $K_\Omega(x, y) = 0$ for all $y \in \partial\Omega$, which (often) eliminates boundary terms when integrating by parts. It has the disadvantage that only in special geometries can it be obtained explicitly, and obtaining estimates on its higher derivatives, which is sometimes necessary, can be difficult, even for simply connected domains. (See, for instance, [1].)

5. Cauchy, Green, and Biot-Savart

To connect our generalizations of Cauchy's theorems to Green's representation theorem, we show in this section how to use the results of Sections 2 and 3 to give an alternate proof of Theorem 4.1.

We continue to work with a vector field $\mathbf{u} \in H_0 \cap H^1(\Omega)^2$. Note that if Ω is simply connected, then H_0 is all of H. Writing $\overleftarrow{\cdot}$ for the inverse of the $\overrightarrow{\cdot}$ map defined in Lemma 1.3, we can apply Theorem 3.1 to $f = \overleftarrow{\overline{\mathbf{u}}}$, so that $\overrightarrow{\overline{f}} = \mathbf{u}$, to obtain, for any $a \in \Omega$,

$$\begin{aligned}
\overleftarrow{\overline{\mathbf{u}}}(a) &= \frac{1}{2\pi i}\int_\gamma \frac{\overleftarrow{\overline{\mathbf{u}}}(z)}{z-a}\, dz - \frac{1}{2\pi i}\int_\Omega \frac{\operatorname{curl}\mathbf{u}(\vec{z}) + i \operatorname{div}\mathbf{u}(\vec{z})}{z-a}\, d\vec{z} \\
&= \frac{1}{2\pi i}\int_\gamma \frac{\overleftarrow{\overline{\mathbf{u}}}(z)}{z-a}\, dz - \frac{1}{2\pi i}\int_\Omega \frac{\operatorname{curl}\mathbf{u}(\vec{z})}{z-a}\, d\vec{z}.
\end{aligned}$$

Note that we could have applied Theorem 3.1 directly to $\overleftarrow{\mathbf{u}}$, but the div and curl operators would not have appeared as neatly.

Taking the conjugate of both sides and converting to vector-form gives

$$\mathbf{u}(a) = \overrightarrow{\frac{i}{2\pi}\int_\gamma \frac{\overleftarrow{\overline{\mathbf{u}}}(z)}{z-a}\, dz} - \overrightarrow{\frac{i}{2\pi}\overline{\int_\Omega \frac{\operatorname{curl}\mathbf{u}(\vec{z})}{z-a}\, d\vec{z}}} = \overrightarrow{\frac{i}{2\pi}\int_\gamma \frac{\overleftarrow{\overline{\mathbf{u}}}(z)}{z-a}\, dz} - \overrightarrow{\frac{i}{2\pi}\int_\Omega \frac{\operatorname{curl}\mathbf{u}(\vec{z})}{\overline{z-a}}}\, d\vec{z}. \tag{5.1}$$

In most applications to the Navier-Stokes equations, we have the classical boundary conditions, $\mathbf{u} = 0$ on $\partial\Omega$. In this special case, the boundary integral vanishes, and we obtain Theorem 5.1.

Theorem 5.1. *Assume that $\mathbf{u} \in H_0$ is a continuously differentiable divergence-free vector field on Ω vanishing on the boundary, and $\omega := \operatorname{curl}\mathbf{u}$ is bounded. Then for all $a \in \Omega$,*

$$\mathbf{u}(\vec{a}) = -\overrightarrow{\frac{i}{2\pi}\int_\Omega \frac{\operatorname{curl}\mathbf{u}(\vec{z})}{\overline{z-a}}}\, d\vec{z}. \tag{5.2}$$

Theorem 5.3 below gives another way to obtain Theorem 5.1, if we assume that $\omega \in L^\infty(\Omega)$, a weaker assumption than $\mathbf{u}$ being continuously differentiable. This method of proof uses the Biot-Savart law of Lemma 5.2 (which is the full-space equivalent of Theorem 4.1).

Lemma 5.2 (Biot-Savart law)**.** *Define the Biot-Savart kernel K on $\mathbb{R}^2$ as in (4.4). If $\omega \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ then $\mathbf{u} = K * \omega$ is the unique bounded divergence-free vector field $\mathbf{u}$ for which $\operatorname{curl}\mathbf{u} = \omega$.*

Theorem 5.3. *Assume that $\mathbf{u} \in H_0 \cap L^\infty(\Omega) \cap H_0^1(\Omega)$ is divergence-free and $\omega := \operatorname{curl} \mathbf{u}$ is bounded. Then for all $a \in \Omega$,*

$$\mathbf{u}(\vec{a}) = K * (\omega \mathbb{1}_\Omega)(\vec{a}) = \overrightarrow{-\frac{i}{2\pi} \int_\Omega \frac{\omega(\vec{z})}{\overline{\overline{z - a}}}\, d\vec{z}}. \tag{5.3}$$

Proof. Because $\mathbf{u} \in H_0^1(\Omega)$, $\mathcal{E}\mathbf{u}$, the extension of $\mathbf{u}$ by zero to all of $\mathbb{R}^2$, lies in $H^1(\mathbb{R}^2)$; moreover, $\mathbf{u}$ also lies in $L^1 \cap L^\infty(\mathbb{R}^2)$. Then $\operatorname{curl} \mathcal{E}\mathbf{u} = \omega$ on Ω while $\operatorname{curl} \mathcal{E}\mathbf{u} = 0$ on $\mathbb{R}^2 \setminus \Omega$. Hence, we can apply the Biot-Savart law to recover $\mathbf{u}$ on all of $\mathbb{R}^2$,

$$\mathbf{u} = K * (\operatorname{curl} \mathcal{E}\mathbf{u}) = K * (\omega \mathbb{1}_\Omega).$$

(Outside of Ω, we obtain 0, so it is only the value on Ω that we care about.)

It remains to show that $K * (\omega \mathbb{1}_\Omega)(\vec{a})$ is the same as the integral on the righthand side of (5.3). This is a matter of unraveling definitions using the $\vec{\cdot}$ operator. We have,

$$-\overrightarrow{\frac{i}{2\pi} \int_\Omega \frac{\omega(\vec{z})}{\overline{\overline{z - a}}}\, dz} = -\frac{1}{2\pi} \int_\Omega \frac{\overrightarrow{i\omega(\vec{z})}}{\overline{\overline{z - a}}}\, d\vec{z}.$$

But,

$$\begin{aligned}
-\frac{1}{2\pi} \frac{\overrightarrow{i\omega(\vec{z})}}{\overline{\overline{z - a}}} &= -\frac{1}{2\pi} \overrightarrow{\frac{\omega(\vec{z})}{|z - a|^2} i(z - a)} = -\frac{1}{2\pi} \frac{\omega(\vec{z})}{|z - a|^2} \overrightarrow{-(z^2 - a^2) + i(z^1 - a^1)} \\
&= -\frac{1}{2\pi} \frac{(\vec{z} - \vec{a})^\perp}{|z - a|^2} \omega(\vec{z}) = K(\overrightarrow{a - z}) \omega(\vec{z}),
\end{aligned}$$

which yields the desired equality. □

We can view Theorem 5.3, then, as a special form of the Biot-Savart law, though with an important caveat: If we use an arbitrary scalar-valued function ω to obtain the vector field $\mathbf{u}$ given by (5.3), that vector field will be divergence-free, but will not, in general, vanish on the boundary nor even be tangential to the boundary.

Moreover, writing (5.3) in the form,

$$\mathbf{u}(x) = \int_\Omega K(x - y) \omega(y)\, dy,$$

we see it is very similar to the expression (4.5). In fact, since the kernel is simpler (and explicit), on its face, it appears superior to that in (4.5). But $K(x - y)$ has no particular value for $y \in \partial\Omega$, which makes it difficult to utilize when integrating by parts, an inevitable occurrence in most PDE applications. So while it is simpler, and estimates on $K(x - y)$ are (much) easier to obtain than estimates on $K_\Omega(x, y)$, it is less effective to utilize, even for solutions to the Navier-Stokes equations, where there is no boundary integral.

Now let us return to (5.1), and assume only that the normal component, $\mathbf{u} \cdot \boldsymbol{n}$, vanishes on the boundary (that is, $\mathbf{u} \in H \cap H^1(\Omega)^2$). The area integral is treated the same way as in Theorems 5.1 and 5.3 and returns a divergence-free vector field, though with no particular value on the boundary. Since $\mathbf{u}$ is divergence-free, the boundary integral must also yield a divergence-free vector field, the combined normal components cancelling to yield zero.

We work now to express the boundary integral in (5.1) in a purely vector-based form. Let

$$A(z) := \frac{\overleftarrow{\mathbf{u}}(z)}{z - a}.$$

Applying Lemma 1.4 gives

$$\begin{aligned}
\frac{i}{2\pi}\overline{\int_\gamma \frac{\overleftarrow{\mathbf{u}}(z)}{z-a}\,dz} &= \frac{i}{2\pi}\overline{\int A(z)\,dz} = \frac{i}{2\pi}\left[\oint_{\partial\Omega} \vec{A}\cdot\boldsymbol{\tau} - i\oint_{\partial\Omega} \vec{A}\cdot\boldsymbol{n}\right] \\
&= \frac{1}{2\pi}\left[\oint_{\partial\Omega} \vec{A}\cdot\boldsymbol{n} + i\oint_{\partial\Omega} \vec{A}\cdot\boldsymbol{\tau}\right] = \frac{1}{2\pi}\oint_{\partial\Omega}\left[\vec{A}\cdot\boldsymbol{n} + i\vec{A}\cdot\boldsymbol{\tau}\right],
\end{aligned}$$

so

$$\overrightarrow{\frac{i}{2\pi}\overline{\int_\gamma \frac{\overleftarrow{\mathbf{u}}(z)}{z-a}\,dz}} = \frac{1}{2\pi}\oint_{\partial\Omega}\left(\vec{A}\cdot\boldsymbol{n},\, \vec{A}\cdot\boldsymbol{\tau}\right).$$

Writing $\mathbf{u} = (u^1, u^2)$ and letting $w = z - a$, we have

$$\begin{aligned}
\overline{A}(z) &= \overline{\left[\frac{\overleftarrow{\mathbf{u}}(z)}{z-a}\right]} = \frac{\overleftarrow{\mathbf{u}}}{\overline{w}} = \frac{\overleftarrow{\mathbf{u}}\,w}{|w|^2} = \frac{(u^1 + iu^2)(w^1 + iw^2)}{|w|^2} \\
&= \frac{u^1w^1 - u^2w^2 + i(u^1w^2 + u^2w^1)}{|w|^2},
\end{aligned}$$

so

$$\vec{A}(z) = \frac{(u^1w^1 - u^2w^2,\, u^1w^2 + u^2w^1)}{|w|^2}.$$

Then,

$$\begin{aligned}
&\left(\vec{A}\cdot\boldsymbol{n},\, \vec{A}\cdot\boldsymbol{\tau}\right) = \left(\vec{A}\cdot(n^1, n^2),\, \vec{A}\cdot(-n^2, n^1)\right) \\
&\quad = \frac{(u^1w^1n^1 - u^2w^2n^1 + u^1w^2n^2 + u^2w^1n^2,\, -u^1w^1n^2 + u^2w^2n^2 + u^1w^2n^1 + u^2w^1n^1)}{|w|^2} \\
&\quad = \frac{(w^1(u^1n^1 + u^2n^2) - u^2w^2n^1 + u^1w^2n^2,\, w^2(u^1n^1 + u^2n^2) - u^1w^1n^2 + u^2w^1n^1)}{|w|^2} \\
&\quad = \frac{(-u^2w^2n^1 + u^1w^2n^2,\, -u^1w^1n^2 + u^2w^1n^1)}{|w|^2} \\
&\quad = \frac{(u^2n^1 - u^1n^2)(-w^2),\, (-u^1n^2 + u^2n^1)w^1}{|w|^2} = \frac{(-(\mathbf{u}\cdot\boldsymbol{\tau})w^2,\, (\mathbf{u}\cdot\boldsymbol{\tau})w^1)}{|w|^2} \\
&\quad = (\mathbf{u}\cdot\boldsymbol{\tau})\frac{\overrightarrow{(z-a)}^\perp}{|z-a|^2}, = -(\mathbf{u}\cdot\boldsymbol{\tau})\frac{\overrightarrow{(a-z)}^\perp}{|a-z|^2},
\end{aligned}$$

where we used $\mathbf{u}\cdot\boldsymbol{n} = 0$. It follows that the boundary term in (5.1) is

$$\frac{1}{2\pi} - \oint_{\partial\Omega} \frac{(\vec{a}-\vec{z})^\perp}{|\vec{a}-\vec{z}|^2}(\mathbf{u}\cdot\boldsymbol{\tau})(\vec{z})\,d\vec{z} = \oint_{\partial\Omega} K(\vec{a}-y)(\mathbf{u}\cdot\boldsymbol{\tau})(y)\,dy. \tag{5.4}$$

This matches the boundary integral in Theorem 4.1. Combined with the treatment in Theorems 5.1 and 5.3 of the area integral, this yields an alternate proof of Theorem 4.1.

Our approach illustrates how Cauchy's integral formula for the value of a function is an expression of the boundary integral in Green's representation formula. It can be most effectively exploited when the underlying function is analytic (meaning as a vector field it is divergence- and curl-free). The power of Cauchy's integral formula is that it allows one to determine the value of a function in a domain given only its value on the domain's boundary (and the strong knowledge that the function is complex-analytic). The Biot-Savart law in a sense works in the opposite direction, taking knowledge of the gradient of a vector field in the domain (via its curl and knowing that it is divergence-free) and giving the value of the function itself in the domain—and so also on the boundary.

6. A generalization Cauchy's integral formula for $n = 2$

When f is analytic, we know from Cauchy's integral theorem that

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^n}\,dz = \frac{f^{(n)}(a)}{n!}.$$

Hence, we should expect to have difficulty extending Theorem 3.1 to generalize this integral, as, at a minimum, we would need a replacement for the complex derivatives of f, which will not exist in general. What we find is the following:

Theorem 6.1. *Let $f\colon \Omega \to \mathbb{C}$ with $\vec{f}$ twice continuously differentiable in a neighborhood of $\overline{\Omega}$. Then for any $a \in \Omega$,*

$$\partial_x f(a) - i\partial_{\overline{z}} f(a) = \frac{1}{2\pi i}\oint_{\partial\Omega} \frac{f(z)}{(z-a)^2}\,dz - \frac{1}{\pi}\,\mathrm{p.\,v.}\int_\Omega \frac{\partial_{\overline{z}} f(z)}{(z-a)^2}\,d\vec{z}. \tag{6.1}$$

The principal value integral (defined in the proof) can be evaluated as

$$\mathrm{p.\,v.}\int_\Omega \frac{\partial_{\overline{z}} f(z)}{(z-a)^2}\,d\vec{z} = \int_\Omega \frac{\partial_{\overline{z}} f(z) - \partial_{\overline{z}} f(a)}{(z-a)^2}\,d\vec{z} + \partial_{\overline{z}} f(a)\int_{\Omega\setminus B_r(x)} \frac{1}{(z-a)^2}\,d\vec{z},$$

where $r > 0$ is arbitrary as long $B_r(x) \subseteq \Omega$.

Moreover, if f is complex-analytic in a neighborhood of $\overline{\Omega}$, the lefthand side of (6.1) is $f'(a)$ and the area integral vanishes (yielding the classical Cauchy's integral formula for $n = 2$).

Remark 6.2. *As we will see in the proof of Theorem 6.1, we could equivalently write*

$$\partial_x f(a) - \frac{1}{2}(\operatorname{curl}\vec{\vec{f}} + i\operatorname{div}\vec{\vec{f}})(\vec{a}) = \frac{1}{2\pi i}\oint_{\partial\Omega} \frac{f(z)}{(z-a)^2}\,dz + \frac{i}{2\pi}\,\mathrm{p.\,v.}\int_\Omega \frac{\operatorname{curl}\vec{\vec{f}}(\vec{z}) + i\operatorname{div}\vec{\vec{f}}(\vec{z})}{(z-a)^2}\,d\vec{z}.$$

Proof of Theorem 6.1. We parallel the proof of Theorem 3.1 as closely as possible, applying Theorem 2.1 to $f(z)/(z-a)^2$. Defining Ω_ε as in the proof of Theorem 3.1, we apply Theorem 2.1 to give

$$\oint_{\partial\Omega_\varepsilon} \frac{f(z)}{(z-a)^2}\,dz = 2i\int_{\Omega_\varepsilon} \partial_{\overline{z}}\frac{f(z)}{(z-a)^2}\,d\vec{z} = 2i\int_{\Omega_\varepsilon} \frac{\partial_{\overline{z}} f(z)}{(z-a)^2}\,d\vec{z}$$

by virtue of (1.3). This holds for all $\varepsilon > 0$ and so holds after taking $\varepsilon \to 0$, giving

$$\begin{aligned}\oint_{\partial\Omega} \frac{f(z)}{(z-a)^2}\,dz &= \lim_{\varepsilon\to 0}\oint_{\partial\Omega_\varepsilon} \frac{f(z)}{(z-a)^2}\,dz + \lim_{\varepsilon\to 0}\oint_{\partial B_\varepsilon(a)} \frac{f(z)}{(z-a)^2}\,dz \\ &= 2i\int_{\Omega_\varepsilon} \frac{\partial_{\overline{z}} f(z)}{(z-a)^2}\,d\vec{z} + \lim_{\varepsilon\to 0}\oint_{\partial B_\varepsilon(a)} \frac{f(z)}{(z-a)^2}\,dz,\end{aligned}$$

if both limits exist.

At this point in the proof of Theorem 3.1, we easily took $\varepsilon \to 0$. Now, however, we need to take much greater care—the integrand in the area integral above is not integrable on all of Ω, and the contour integral, on the face of it, seems to scale like $C\varepsilon^{-1}$.

Let $F(\vec{z}) := \partial_{\overline{z}} f(z)$. Then,

$$\int_{\Omega_\varepsilon} \frac{\partial_{\overline{z}} f(z)}{(z-a)^2}\,d\vec{z} = \int_{\Omega_\varepsilon} \frac{F(\vec{z}) - F(\vec{a})}{(z-a)^2}\,d\vec{z} + F(\vec{a})\int_{\Omega_\varepsilon} \frac{1}{(z-a)^2}\,d\vec{z}.$$

Let $B_r(a)$ be any ball of radius $r > 0$ such that $B_r(a) \subseteq \Omega$. Then

$$\frac{1}{(z-a)^2} = \frac{(\overline{z-a})^2}{|z-a|^4} = \frac{(x-a^1)^2 - (y-a^2)^2 - 2i(x-a^1)(y-a^2)}{|z-a|^4},$$

and we can see by symmetry that

$$\int_{B_r \setminus B\varepsilon} \frac{1}{(z-a)^2} \, d\vec{z} = 0. \tag{6.2}$$

Moreover, ∇F is bounded because f is twice continuously differentiable. Thus,

$$\left| \frac{F(\vec{z}) - F(\vec{a})}{(z-a)^2} \right| \le \left\| \frac{F(\vec{\cdot}) - F(\vec{a})}{\cdot - a} \right\|_{L^\infty} \frac{1}{|z-a|} \le \frac{\|\nabla F\|_{L^\infty}}{|z-a|}.$$

This shows that $\frac{F(\vec{z})-F(\vec{a})}{(z-a)^2}$ is locally integrable and that we can take $\varepsilon \to 0$ to obtain

$$\int_\Omega \frac{\partial_{\overline{z}} f(z)}{(z-a)^2} \, d\vec{z} = \int_\Omega \frac{F(\vec{z}) - F(\vec{a})}{(z-a)^2} \, d\vec{z} + F(\vec{a}) \text{ p. v.} \int_\Omega \frac{1}{(z-a)^2} \, d\vec{z}.$$

Here, p. v. means a specific type of principal value integral:

$$\text{p. v.} \int_\Omega g(z) \, d\vec{z} = \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} g(z) \, d\vec{z}$$

if it exists. For $g(z) = (z-a)^{-2}$ we can be a little more explicit. From (6.2), we see that

$$\text{p. v.} \int_\Omega \frac{1}{(z-a)^2} \, d\vec{z} = \int_{\Omega \setminus B_r(x)} \frac{1}{(z-a)^2} \, d\vec{z},$$

where we can choose $r > 0$ so that $B_r(x)$ is the largest ball lying in Ω.

This leaves the boundary, integral,

$$\begin{aligned} \oint_{\partial B_\varepsilon(a)} \frac{f(z)}{(z-a)^2} \, dz &= \oint_{\partial B_\varepsilon(a)} \frac{f(z) - f(a)}{(z-a)^2} \, dz + f(a) \oint_{\partial B_\varepsilon(a)} \frac{dz}{(z-a)^2} \\ &= \oint_{\partial B_\varepsilon(a)} \frac{f(z) - f(a)}{(z-a)^2} \, dz, \end{aligned}$$

where we applied Cauchy's integral theorem.

Now, we assumed that f is twice continuously differentiable, which means we can apply Taylor's theorem to conclude that

$$\overrightarrow{f(z) - f(a)} = A\overrightarrow{(z-a)} + \overrightarrow{R(z)},$$

where

$$A := \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} \tag{6.3}$$

evaluated at $\vec{a}$ and $|R(z)| \le M|z-a|^2$ for all z sufficiently close to a. Then,

$$\lim_{\varepsilon \to 0} \left| \oint_{\partial B_\varepsilon(a)} \frac{R(z)}{(z-a)^2} \, dz \right| \le \lim_{\varepsilon \to 0} 2\pi\varepsilon M = 0,$$

so

$$\lim_{\varepsilon \to 0} \oint_{\partial B_\varepsilon(a)} \frac{f(z)}{(z-a)^2} \, dz = \lim_{\varepsilon \to 0} \oint_{\partial B_\varepsilon(a)} \frac{\left[\overleftarrow{A\overrightarrow{(z-a)}} \right]}{(z-a)^2} \, dz.$$

(We know this limit exists because the original integral exists and we showed the limit above exists, but we also show its existence independently in what follows.)

In the integrand, we can let $z - a = \varepsilon e^{i\theta}$ for $\theta \in [0, 2\pi]$. We can then write the integral as

$$\begin{aligned}\oint_{\partial B_\varepsilon(a)} \frac{\left[\overleftarrow{A\overrightarrow{(z-a)}}\right]}{(z-a)^2}\, dz &= \frac{1}{\varepsilon}\int_0^{2\pi} \frac{A(\cos\theta, \sin\theta)}{e^{2i\theta}} i\varepsilon e^{i\theta}\, d\theta \\ &= i\int_0^{2\pi} \left(A_1^1 \cos\theta + A_2^1 \sin\theta + i(A_1^2\cos\theta + A_2^2 \sin\theta)\right) e^{-i\theta}\, d\theta.\end{aligned}$$

Writing,

$$\begin{aligned}A_1^1 \cos\theta + &A_2^1 \sin\theta + i(A_1^2 \cos\theta + A_2^2 \sin\theta) \\ &= A_1^1 e^{i\theta} + iA_1^2 e^{i\theta} + (A_2^1 + A_1^2)\sin\theta + i(A_2^2 - A_1^1)\sin\theta,\end{aligned}$$

we see that

$$\begin{aligned}\oint_{\partial B_\varepsilon(a)} \frac{\left[\overleftarrow{A\overrightarrow{(z-a)}}\right]}{(z-a)^2}\, dz &= i\int_0^{2\pi} (A_1^1 + iA_1^2)\, d\theta + i\int_0^{2\pi} (A_2^1 + A_1^2 + i(A_2^2 - A_1^1)) \sin\theta e^{i\theta}\, d\theta \\ &= 2\pi i(A_1^1 + iA_1^2) + i(A_2^1 + A_1^2 + i(A_2^2 - A_1^1)) \int_0^{2\pi} (\sin\theta\cos\theta + i\sin^2\theta)\, d\theta \\ &= 2\pi i(A_1^1 + iA_1^2) + \pi i(A_2^1 + A_1^2 + i(A_2^2 - A_1^1)) \\ &= \pi i(2A_1^1 + i2A_1^2 + A_2^1 + A_1^2 + i(A_2^2 - A_1^1)) \\ &= \pi i(2A_1^1 + A_2^1 + A_1^2 + i(2A_1^2 + A_2^2 - A_1^1)).\end{aligned}$$

But,

$$\begin{aligned}2A_1^1 + &A_2^1 + A_1^2 + i(2A_1^2 + A_2^2 - A_1^1) = 2\partial_x u + \partial_y u + \partial_x v + i(2\partial_x v + \partial_y v - \partial_x u) \\ &= 2(\partial_x u + i\partial_x v) - (\operatorname{curl} \vec{\vec{f}} + i \operatorname{div} \vec{\vec{f}}) = 2\partial_x f - (\operatorname{curl} \vec{\vec{f}} + i \operatorname{div} \vec{\vec{f}}).\end{aligned}$$

Also,

$$\begin{aligned}2A_1^1 + &A_2^1 + A_1^2 + i(2A_1^2 + A_2^2 - A_1^1) = 2(\partial_x u + i\partial_x v) + \partial_y u + \partial_x v + i(\partial_y v - \partial_x u) \\ &= 2\partial_x f + \partial_y(u + iv) + \partial_x(v - iu) = 2\partial_x f + \partial_y f - i\partial_x f = 2\partial_x f - 2i\partial_{\bar z} f.\end{aligned}$$

In particular, if f is analytic, then $\partial_{\bar z f} = 0$ and $\partial_x f = f'$, so

$$2A_1^1 + A_2^1 + A_1^2 + i(2A_1^2 + A_2^2 - A_1^1) = 2f'(a).$$

Putting this all together, we obtain the result. □

Consider the form of Theorem 6.1 given in Remark 6.2. Viewed in terms of vector fields (without explicitly making the translation), it tells us that if we know the value of a vector field on the boundary along with its div and curl everywhere, we can obtain its full gradient, the matrix A of (6.3): Remark 6.2 gives us $\partial_x u$ and $\partial_y u$, and then $\partial_x v = \operatorname{curl} f + \partial_y u$, $\partial_y v = \operatorname{div} f - \partial_x u$.

In another connection with incompressible fluid mechanics, if $f = \overleftarrow{\mathbf{u}}$, where $\mathbf{u}$ is an incompressible vector field, the expression in Remark 6.2 is the analog of the formula relating $\operatorname{curl} \mathbf{u}$ (the *vorticity*) to the velocity gradient $\nabla \mathbf{u}$ via a singular integral operator whose kernel is ∇K, K being the Biot-Savart kernel of (4.4). See, for instance, Proposition 2.20 of [3] for the whole-plane result.

References

[1] David M. Ambrose, James P. Kelliher, Milton C. Lopes Filho, and Helena J. Nussenzveig Lopes. Serfati solutions to the 2D Euler equations on exterior domains. *J. Differential Equations*, 259(9):4509–4560, 2015. 8

[2] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010. 7

[3] Andrew J. Majda and Andrea L. Bertozzi. *Vorticity and incompressible flow*, volume 27 of *Cambridge Texts in Applied Mathematics*. Cambridge University Press, Cambridge, 2002. 13

[4] D. Pompeiu. Sur la continuité des fonctions de variables complexes. *Annales de la faculté des sciences de Toulouse* 2^e *série*, 7(3):265–315, 1905. 5

[1] Department of Mathematics, University of California, Riverside, 900 University Ave., Riverside, CA 92521, U.S.A.

Email address: `kelliher@math.ucr.edu`