# EIGENVALUES OF THE STOKES OPERATOR VERSUS THE DIRICHLET LAPLACIAN IN THE PLANE

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Abstract. We show that the *k-th* eigenvalue of the Dirichlet Laplacian is strictly less than the *k-th* eigenvalue of the classical Stokes operator (equivalently, of the clamped buckling plate problem) for a bounded domain in the plane having a locally Lipschitz boundary. For a $C^2$ boundary, we show that eigenvalues of the Stokes operator with Navier slip (friction) boundary conditions interpolate continuously between eigenvalues of the Dirichlet Laplacian and of the classical Stokes operator.



## 1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with locally Lipschitz boundary $\Gamma$. Let $\sigma_D$ be the spectrum of the negative Laplacian with homogenous Dirichlet boundary conditions (which we refer to as the *Dirichlet Laplacian*) and let $\sigma_S$ be the spectrum of the Stokes operator with homogenous Dirichlet boundary conditions (which we refer to as the *classical Stokes operator*). Equivalently, $\sigma_S$ is the set of eigenvalues of the clamped buckling plate problem ([23], [24], [9]). Each spectrum is discrete with

$$\sigma_D = \{\lambda_j\}_{j=1}^\infty, \; 0 < \lambda_1 < \lambda_2 \le \cdots, \tag{1.1}$$

$$\sigma_S = \{\nu_j\}_{j=1}^\infty, \; 0 < \nu_1 \le \nu_2 \le \cdots, \tag{1.2}$$

each eigenvalue repeated according to its multiplicity.

We prove the following:

**Theorem 1.1.** *For all positive integers $k$, $\lambda_k < \nu_k$.*

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Further, let $\gamma_k(\theta)$ be the $k$-*th* eigenvalue of the Stokes operator with boundary conditions $(1-\theta)\omega(u)+\theta u\cdot\boldsymbol{\tau} = u\cdot\mathbf{n} = 0$, where $\omega(u) = \partial_1 u^2 - \partial_2 u^1$ is the vorticity of $u$ and $\boldsymbol{\tau}, \mathbf{n}$ are the tangential, normal unit vectors (see Section 8 for details). We also prove:

**Theorem 1.2.** *When $\Gamma$ is $C^2$ and has a finite number of components, for all positive integers $k$, the function $\gamma_k$ is a strictly increasing continuous bijection from $[0,1]$ onto $[\lambda_k, \nu_k]$.*

Theorem 1.1 is the analog of the inequality $\mu_{k+1} < \lambda_k$ for $k = 1, 2, \dots$ proved by Filonov in [6]. Here, $\sigma_N = \{\mu_j\}_{j=1}^\infty$ is the spectrum of the negative Laplacian with homogenous Neumann boundary conditions (which we refer to as the *Neumann Laplacian*). Then $\sigma_N$ is also discrete with $0 = \mu_1 < \mu_2 \le \cdots$. Filonov's inequality applies in $\mathbb{R}^d$, $d \ge 2$ and only requires that $\Omega$ have finite measure and that its boundary have sufficient regularity that the embedding of $W^1(\Omega)$ in $L^2(\Omega)$ is compact, which is slightly weaker than our assumption that $\Gamma$ is locally Lipschitz. Because of the need to integrate by parts, however, we require the additional regularity.

Filonov's strict inequality is a strengthening of the partial inequality, $\mu_{k+1} \le \lambda_k$, proved by L. Friedlander in [8] using very different techniques.

A fairly direct variational argument gives $\lambda_k \le \nu_k$ (see Remark 5.3 or Equation (1.8) of [2]); it is the strict inequality in Theorem 1.1 that is of interest.

For the unit disk, where one can calculate the eigenfunctions explicitly,

$$\sigma_D = \left\{ j_{nk}^2 : n = 0, 1 \dots, k = 1, 2, \dots \right\},$$
$$\sigma_S = \left\{ j_{nk}^2 : n = 1, 2 \dots, k = 1, 2, \dots \right\},$$

where $j_{nk}$ is the $k$-*th* positive zero of the Bessel function $J_n$ of the first kind of order $n$. Each eigenvalue has multiplicity 2 except for $\left\{ j_{0k}^2 : k \in \mathbb{N} \right\} \subseteq \sigma_D$ and $\left\{ j_{1k}^2 : k \in \mathbb{N} \right\} \subseteq \sigma_S$, which have multiplicity 1. This gives the ordering $0 < \lambda_1 < \lambda_2 = \lambda_3 = \nu_1 < \lambda_4 = \lambda_5 = \nu_2 = \nu_3 < \lambda_6 < \cdots$. In this case we have $\lambda_{k+1} \le \nu_k$ for all $k$ but $\lambda_{k+1} \not< \nu_k$ for $k = 1$. This leaves open the possibility that $\lambda_{k+1} \le \nu_k$ in full generality. This inequality was conjectured to hold by L. E. Payne many years ago, but has remained unproved.

To prove Theorem 1.1 we adapt Filonov's proof in [6] that $\mu_{k+1} < \lambda_k$, which is shockingly direct and simple. As we observed for a disk, $\lambda_{k+1} \not< \nu_k$, which shows that some aspect of Filonov's approach must fail if we attempt to adapt it to obtain Theorem 1.1. In fact, what fails is his use of a function of the form $f = e^{i\omega\cdot x}$ with $|\omega|^2 = \lambda$ for $\lambda > 0$, which has the properties that $\Delta f + \lambda f = 0$ and $|\nabla f|^2 = \lambda |f|$. This serves as an "extra" function that increases the dimension of a subspace of functions that he shows satisfy the bound in the variational formulation of the eigenvalue problem for the Neumann Laplacian. There can be no such function that will serve in general for us (else $\lambda_{k+1} < \nu_k$ would hold in general), but we describe the analog of such a function in our setting in Section 7, show that given its existence we obtain $\lambda_{k+1} \le \nu_k$, and explain why it fails to give $\lambda_{k+1} < \nu_k$.

Our proof of $\lambda_k < \nu_k$ is largely a matter of transforming the eigenvalue problems so that the Stokes operator can play the role the Dirichlet Laplacian plays for Filonov and so the Dirichlet Laplacian can play the role that the Neumann Laplacian plays for Filonov.

The approach of Friedlander in [8] can also be adapted to prove Theorem 1.1, at least for $C^1$-boundaries.

In Section 8 we show that when $\Gamma$ is $C^2$ and has a finite number of components, one can interpolate continuously between $\lambda_j$ and $\nu_j$ using the eigenvalues of the negative Laplacian with Navier slip boundary conditions (Theorem 1.2). These boundary conditions, originally defined by Navier, have recently received considerable attention among fluid mechanics as a physically motivated replacement for Dirichlet boundary conditions, as they allow a thorough characterization of the boundary layer. See, for instance, [4], [21], [17], [15], and [16]. We also discuss Neumann boundary conditions for the velocity and for the vorticity, and Robin boundary conditions for the vorticity.

This paper is organized as follows: We describe the necessary function spaces, trace operators, and related lemmas in Section 2. In Section 3 we define the classical Stokes operator and a variant of it using Lions boundary conditions (vanishing vorticity on the boundary). We show that the eigenvalue problem for the classical Stokes operator is equivalent to the eigenvalue problem for the clamped buckling plate problem. We also describe the strong forms of the associated eigenvalue problems in Section 3, giving the weak forms in Section 4. In Section 5 we describe the variational (min-max) formulations of the eigenvalue problems, using these formulations in Section 6 to prove Theorem 1.1. In Section 7, we describe the properties of the analog of the function $f$ used by Friedlander and Filonov and prove that its existence would imply the inequality $\lambda_{k+1} \le \nu_k$. Finally, in Section 8 we discuss Navier boundary conditions and prove Theorem 1.2.

For a vector field $u$ we define $u^\perp = (-u^2, u^1)$ and for a scalar field $\psi$ we define $\nabla^\perp \psi = (-\partial_2 \psi, \partial_1 \psi)$. Observe that $(u^\perp)^\perp = -u$ and $(\nabla^\perp)^\perp \psi = -\nabla\psi$. By $\omega(u)$ we mean the vorticity (scalar curl) of $u$: $\omega(u) = \partial_1 u^2 - \partial_2 u^1$. We make frequent use of the identities $\nabla^\perp \omega(u) = \Delta u$ and $\omega(u) = -\operatorname{div} u^\perp$, the former requiring that $u$ be divergence-free.

**We assume throughout that $\Omega$ is a bounded domain whose boundary, $\Gamma$, unless specifically stated otherwise, is locally Lipschitz.**

## 2. Function spaces and related facts

Let $\mathbf{n}$ be the outward-directed unit normal vector to $\Gamma$ and $\boldsymbol{\tau}$ be the unit tangent vector chosen so that $(\mathbf{n}, \boldsymbol{\tau})$ has the same orientation as the Cartesian unit vectors $(\mathbf{e}_1, \mathbf{e}_2)$. These vectors are defined almost everywhere on $\Gamma$ since $\Gamma$ is locally Lipschitz.

The spaces $C^{k,\alpha}(\Omega)$, $C^{k,\alpha}(\overline{\Omega})$, and $W^s(\Omega)$ are the usual Hölder and $L^2$-based Sobolev spaces, $k$ an integer, $0 \le \alpha \le 1$, and $s$ any real number.

About these spaces, which can be defined in various equivalent ways, we need to say a few words.

Defining the norms,

$$\|f\|_{C^k} = \sum_{j=0}^{m} \sup_{\Omega} \sup_{|\beta|=j} \left| D^\beta u \right|,$$

$$\|f\|_{C^{k,\alpha}} = \|f\|_{C^k} + \sup_{|\beta|=k} \sup_{x \neq y \in \Omega} \frac{\left| D^\beta f(x) - D^\beta f(y) \right|}{|x-y|^\alpha},$$

$0 < \alpha \le 1$, $C^k(\Omega) = C^{k,0}(\Omega)$ and $C^{k,\alpha}(\Omega)$ are the spaces of functions finite under their respective norms; $C^{k,\alpha}(\overline{\Omega})$ is defined similarly. Here $\beta$ is a multi-index.

When $m \ge 0$ is an integer, $W^m(\Omega)$ is the completion of the space of all $C^\in(\Omega)$-functions in the norm,

$$\|f\|_{W^m} = \left( \sum_{|\alpha| \le m} \|D^\alpha f\|^2_{L^2(\Omega)} \right)^{1/2},$$

where $\alpha$ is a multi-index. Equivalently, $W^m(\Omega)$ is the space of all functions $f$ such that $D^\alpha f$ is in $L^2(\Omega)$ for all $|\alpha| \le m$. $W^m_0(\Omega)$ is defined similarly as the closure of $C^\infty_0(\Omega)$ under the $W^m$-norm. (See, for instance, Section 3.1 of [1].) $W^1_0(\Omega)$ can equivalently be defined as all functions in $W^1(\Omega)$ whose boundary trace is zero. $W^{-m}(\Omega)$ is the dual space of $W^m_0(\Omega)$. Fractional Sobolev spaces, $W^s(\Omega)$ can be defined, for instance, as in Theorem 7.48 of [1].

On $\Omega$, we will only need integer-order Hölder and Sobolev spaces, but on $\Gamma$ we will need to use fractional spaces. Hölder spaces, however, can only be defined when the boundary has sufficient regularity.

We define a bounded domain $\Omega$ (or its boundary $\partial\Omega$) to be of class $C^{k,\alpha}$, $k \ge 0$ an integer, $0 \le \alpha \le 1$, if locally there exists a $C^{k,\alpha}$ diffeomorphism $\psi$ that maps $\Omega$ into the upper half-plane with $\partial\Omega$ being mapped to an open interval $I$. We say that $\varphi$ is in $C^{k,\alpha}(\partial\Omega)$ if $\varphi \circ \psi^{-1}$ is in $C^{k,\alpha}(I)$. We also write $C^k$ for $C^{k,0}$. If $\Omega$ is a $C^{k,\alpha}$ domain and $\varphi$ lies in $C^{j,\beta}(\partial\Omega)$ for $j + \beta \le k + \alpha$ then there exists an extension of $\varphi$ to $C^{j,\beta}(\overline{\Omega})$. See Section 6.2 of [12] for more details. The inverse operation of restricting to the boundary gives an equivalent definition of $C^{k,\alpha}(\partial\Omega)$ as restrictions of functions in $C^{k,\alpha}(\overline{\Omega})$.

When $\Gamma$ is locally Lipschitz, we will only have need for $W^s(\partial\Omega)$ for $s = \pm 1/2$ and 0. We define $W^{1/2}(\partial\Omega)$ to be the image (a subspace of $L^2(\partial\Omega)$) under the unique continuous extension to $W^1(\Omega)$ of the map that restricts the value of a $C^\infty(\overline{\Omega})$-function to the boundary. The existence of this extension was proven by Gagliardo [10] (or see Theorem 1.5.1.3 of [13]). Alternately, we could define $W^{1/2}(\Omega)$ intrinsically as in Section II.3 of [11]. We define $W^{-1/2}(\partial\Omega)$ to be the dual space to $W^{1/2}(\partial\Omega)$ and let $W^0(\partial\Omega) = L^2(\partial\Omega)$.

For $C^2$ boundaries, we will need Corollary 2.2 and hence need to define $W^s(\partial\Omega)$ for all real $s$. We use the intrinsic definition of $W^s(\partial\Omega)$ due to J. L. Lions, which applies when the boundary is of class $C^m$, $m \geq 1$. This definition is similar to that for the Hölder spaces defined above, and requires for $s > 0$ that each $\varphi \circ \psi^{-1}$ be of class $W^s(I)$, where $I$ is the domain of $\psi^{-1}$. (See p. 215-217 of [1] for more details.) For $s < 0$ we define $W^s(\partial\Omega)$ to be the dual space of $W^{-s}(\partial\Omega)$ and let $W^0(\partial\Omega) = L^2(\partial\Omega)$ as above. It follows from Theorem 7.53 of [1] that the two definitions of these spaces are equivalent for $0 < s \leq m$ and hence for all real $s$. (Adams gives the proof only for $s = m - 1/2$, from which it follows immediately for all $s = j - 1/2$, $j$ an integer with $1 \leq j \leq m$, since if $\partial\Omega$ is of class $C^m$ it is of class $C^k$ for all $1 \leq k \leq m$. We only need the equivalence for $m = 2$, $s = 1/2$, so this will suffice.)

**Lemma 2.1.** *Let $D$ be any bounded domain in $\mathbb{R}^n$ with $C^\infty$ boundary. Let $\varphi$ lie in $C^{k,\alpha}(\overline{D})$ and $f$ lie in $W^s(D)$, $s > 0$. Then $\varphi f$ lies in $W^s(D)$ as long as*

$$\begin{cases} k + \alpha \geq s, & s \text{ an integer}, \\ k + \alpha > s, & s \text{ not an integer}. \end{cases}$$

*Let $g$ lie in $W^{s'}(D)$. Then $fg$ lies in $W^s(D)$ if $s' > s$ and $s' \geq n/2$ or if $s' \geq s$ and $s' > n/2$.*

*Proof.* This follows from Theorems 1.4.1.1 and 1.4.4.2 of [11]. □

**Corollary 2.2.** *Assume that $\Gamma$ is of class $C^{k,\alpha}$. Then for all $\varphi$ in $C^{j,\beta}(\partial\Omega)$ for $j + \beta \leq k + \alpha$ and $f$ in $W^s(\Gamma)$ for $s > 0$, $\varphi f$ lies in $W^s(\Gamma)$ as long as*

$$\begin{cases} j + \beta \geq s, & s \text{ an integer}, \\ j + \beta > s, & s \text{ not an integer}. \end{cases}$$

*If $f$ lies in $W^s(\Gamma)$ and $\varphi$ lies in $W^{s+\epsilon}(\Gamma)$, $\epsilon > 0$, then $\varphi f$ lies in $W^s(\Gamma)$ if $s \geq 1/2$.*

*Proof.* Apply Lemma 2.1 to the functions $\varphi \circ \psi^{-1}$ and $f \circ \psi^{-1}$ with domain $D = I$. □

**Corollary 2.3.** *Assume that $\Gamma$ is $C^2$. Then $g\boldsymbol{\tau}$ and $g\mathbf{n}$ are in $W^{1/2}(\Gamma)$ for any $g$ in $W^{1/2}(\Gamma)$, and $u\cdot\boldsymbol{\tau}$ and $u\cdot\mathbf{n}$ are in $W^{1/2}(\Gamma)$ for any $u$ in $(W^{1/2}(\Gamma))^2$.*

*Proof.* Because $\Gamma$ is $C^2$, $\boldsymbol{\tau}$ and $\mathbf{n}$ are in $C^1 = C^{1,0}$. But $1 + 0 > 1/2$, so the second condition in Corollary 2.2 applies in each case to give the result. □

Let

$$\mathcal{V} = \left\{u \in (C_0^\infty(\Omega))^2 : \operatorname{div} u = 0\right\}$$

be the space of complex vector-valued divergence-free test functions on $\Omega$. We let $H$ be the completion of $\mathcal{V}$ in $L^2(\Omega)$ and $V$ be the completion of $\mathcal{V}$ in

$W_0^1(\Omega)$. These definitions of $H$ and $V$ are valid for arbitrary domains. We will also find use for the space

$$E(\Omega) = \left\{v \in (L^2(\Omega))^2 \colon \operatorname{div} v \in L^2(\Omega)\right\} \tag{2.1}$$

with $\|u\|_{E(\Omega)} = \|u\|_{L^2(\Omega)} + \|\operatorname{div} u\|_{L^2(\Omega)}$.

We use $(\cdot,\cdot)$ to mean the inner product in $L^2(\Omega)$: $(u, v) = \int_\Omega u\overline{v}$ or sometimes to mean the pairing of $v$ in a space $Z$ with $u$ in $Z^*$ or of $v$ in $\mathcal{D}(\Omega)$ with $u$ in $\mathcal{D}'(\Omega)$: which is meant is stated if it is not clear from context.

The various integrations by parts that we will make are justified by Lemma 2.4, which is Theorem 1.2 p. 7 of [25] for locally Lipschitz domains. (Temam states the theorem for $C^2$ boundaries but the proof for locally Lipschitz boundaries is the same, using a trace operator for Lipschitz boundaries in place of that for $C^2$ boundaries: see p. 117-119 of [11], in particular, Theorem 2.1 p. 119.)

**Lemma 2.4.** *There exists an extension of the trace operator $\gamma_{\mathbf{n}} \colon (C_0^\infty(\overline{\Omega}))^2 \to C^\infty(\Gamma)$ defined by $u \mapsto u \cdot \mathbf{n}$ on $\Gamma$ to a continuous linear operator from $E(\Omega)$ onto $W^{-1/2}(\Gamma)$. The kernel of $\gamma_{\mathbf{n}}$ is the space $E_0(\Omega)$—the completion of $C_0^\infty(\Omega)$ in the $E(\Omega)$ norm. For all $u$ in $E(\Omega)$ and $f$ in $W^1(\Omega)$,*

$$(u, \nabla f) + (\operatorname{div} u, f) = \int_\Gamma (u \cdot \mathbf{n})\overline{f}. \tag{2.2}$$

**Remark 2.5.** In Equation (2.2) and in what follows we usually do not explicitly include the trace operators. On the right-hand side of Equation (2.2), for instance, $u \cdot \mathbf{n}$ is actually $\gamma_{\mathbf{n}} u$, which is thus in $W^{-1/2}(\Gamma)$, and $f$ is actually $\gamma_0 f$, where $\gamma_0$ is the usual trace operator from $W^s(\Omega)$ to $W^{s-1/2}(\Gamma)$ for all $s > 1/2$. Also, the boundary integral should more properly be written as a pairing in the duality between $W^{-1/2}(\Gamma)$ and $W^{1/2}(\Gamma)$ of $u \cdot \mathbf{n}$ and $f$.

**Lemma 2.6.** *$W^s(\Omega)$ is compactly embedded in $W^r(\Omega)$ for all $s > r \geq 0$.*

*Proof.* This is an instance of the Rellich-Kondrachov theorem. That it holds for a bounded domain with locally Lipschitz boundary follows, for instance, from the comments on p. 67 and Theorem 6.2 p. 144 of [1]. □

We will use several times the following basic result of elliptic regularity theory:

**Lemma 2.7.** *Let $f$ lie in $W^{-1}(\Omega)$. There exists a unique $\psi$ in $W_0^1(\Omega)$ that is a weak solution of $\Delta\psi = f$. Furthermore, $\|\psi\|_{W^1(\Omega)} \leq C\,\|f\|_{W^{-1}(\Omega)}$. When $\Gamma$ is $C^2$ and $f$ is in $L^2(\Omega)$, $\|\psi\|_{W^2(\Omega)} \leq C\,\|\Delta\psi\|_{L^2(\Omega)}$.*

*Proof.* See, for instance, p. 118-121 of [18] for general bounded open domains and Theorem 4 of [5] and the remark following it on p. 317 for $C^2$ boundaries. □

Poincare's inequality holds in both its classical forms:

**Lemma 2.8.** *Let $f$ lie in $W_0^1(\Omega)$ or else lie in $W^1(\Omega)$ with $\int_\Omega f = 0$. Then there exists a constant $C$ such that*

$$\|f\|_{L^2(\Omega)} \le C \|\nabla f\|_{L^2(\Omega)}.$$

*Proof.* See Theorem 4.1 p. 49 and Theorem 4.3 p. 54 of [11]. □

Since $\Gamma$ is locally Lipschitzian, we can define

$$\widehat{H} = \left\{u \in (L^2(\Omega))^2 : \operatorname{div} u = 0 \text{ in } \Omega,\, \gamma_{\mathbf{n}} u = 0 \text{ on } \Gamma\right\},$$
$$\widehat{V} = \left\{u \in (W^1(\Omega))^2 : \operatorname{div} u = 0 \text{ in } \Omega,\, \gamma_0 u = 0 \text{ on } \Gamma\right\}.$$

By the continuity of the trace operators $\gamma_{\mathbf{n}}$ and $\gamma_0$ it follows that $H \subseteq \widehat{H}$ and $V \subseteq \widehat{V}$. When $\Gamma$ is a bounded domain with locally Lipschitz boundary, $H = \widehat{H}$ and $V = \widehat{V}$. For $H = \widehat{H}$ see Theorem 1.4 Chapter 1 of [25]. That $V = \widehat{V}$ is proved in [22] (see comments p. 148 of [11] and p. 67 of [1]).

**Lemma 2.9.** *Assume that $u$ is in $(\mathcal{D}'(\Omega))^2$ with $(u, v) = 0$ for all $v$ in $\mathcal{V}$. Then $u = \nabla p$ for some $p$ in $\mathcal{D}'(\Omega)$. If $u$ is in $(L^2(\Omega))^2$ then $p$ is in $W^1(\Omega)$; if $u$ is in $H$ then $p$ is in $W^1(\Omega)$ and $\Delta p = 0$.*

*Proof.* For $u$ in $(\mathcal{D}'(\Omega))^2$ see Proposition 1.1 p. 10 of [25]. For $u$ in $(L^2(\Omega))^2$ the result follows from a combination of Theorem 1.1 p. 103 and Remark 4.1 p. 54 of [11] (also see Remark 1.4 p. 11 of [25]). □

We will also find a need for the following spaces:

$$Y = Y^1 = H \cap W^1(\Omega),\ Y^2 = \left\{u \in Y \colon \omega \in W^1(\Omega)\right\},$$
$$Y_0^2 = \left\{u \in Y \colon \omega(u) \in W_0^1\right\},$$
$$X = X^1 = \left\{u \in H \colon \omega(u) \in L^2(\Omega)\right\},\ X^2 = \left\{u \in H \colon \omega(u) \in W^1\right\},$$
$$X_0^2 = \left\{u \in H \colon \omega(u) \in W_0^1\right\},$$

with the obvious norms on each space. We give $Y$ the $W^1(\Omega)$ norm, but place no norm on the other spaces. When $\Gamma$ is $C^2$ and has a finite number of components, the $X$ and $Y$ spaces coincide as in Corollary 2.16.

The average value of any vector $u$ in $H$—and hence in all of our spaces—is zero, as can be seen by integrating $u \cdot e_i$ over $\Omega$, where $e_i = \nabla x_i$ is a coordinate vector, and applying Lemma 2.4. Thus, Poincare's inequality holds for $Y$ and $V$ so we can, and will, use

$$\|u\|_Y = \|u\|_V = \|\nabla u\|_{L^2(\Omega)}$$

in place of the $W^1(\Omega)$ norm for these two spaces.

Let

$$H_c = \{v \in H \colon \omega(v) = 0\}$$

and, noting that $H_c$ is a closed subspace of $H$, let $W_0$ be the orthogonal complement of $H_c$ in $H$. Thus, $H = W_0 \oplus H_c$ is an orthogonal decomposition of $H$. Observe that $V \cap W_0 = V$, and when $\Omega$ is simply connected, $H = W_0$.

**Lemma 2.10.** *For any $u$ in $W_0$ there exists a stream function $\psi$ in $W^1(\Omega)$ for $u$—that is, $u = \nabla^\perp \psi$—and $\psi$ is unique up to the addition of a constant. Moreover,*

$$W_0 = \left\{\nabla^\perp \psi \colon \psi \in W_0^1(\Omega)\right\} = \nabla^\perp W_0^1(\Omega).$$

*If $u$ is in $W_0 \cap Y$ then $\psi$ can be taken to lie in $W_0^1(\Omega) \cap W^2(\Omega)$ and if $u$ is in $V$ then $\psi$ can be taken to lie in $W_0^2(\Omega)$.*

*Proof.* Let $u$ be in $W_0$, and let $\psi$ in $W_0^1(\Omega)$ solve $\Delta\psi = \omega(u) \in W^{-1}(\Omega)$ as in Lemma 2.7. Letting $w = \nabla^\perp\psi \in L^2(\Omega)$, we have $\omega(w) = \Delta\psi = \omega(u)$, $\operatorname{div} w = 0$, and $w \cdot \mathbf{n} = 0$ on $\Gamma$, so $w$ is in $H$. Thus, $w$ is a vector in $H$ with the same vorticity as $u$, meaning that $u - w$ is in $H_c$.

We claim that $w$ is in $W_0$. To see this, let $v$ be in $H_c$. Then

$$(w, v) = (\nabla^\perp\psi, v) = (-\nabla\psi, v^\perp) = (\psi, \operatorname{div} v^\perp) + \int_\Gamma (v^\perp \cdot \mathbf{n})\psi = 0.$$

The last equality follows from $\operatorname{div} v^\perp = \omega(v) = 0$ (showing also that $v^\perp$ is in $E(\Omega)$ and allowing integration by parts via Lemma 2.4) and $\psi = 0$ on $\Gamma$. Since this is true for all $v$ in $H_c$, it follows that $w$ is in $W_0$.

Thus, both $u$ and $w$ are in $W_0$, so $u - w$ is in $W_0$. But we already saw that $u - w$ is in $H_c$, so $u - w = 0$.

What we have shown is both the existence of a stream function and the expression for $W_0$, the uniqueness of the stream function up to a constant being then immediate. The additional regularity of $\psi$ for $u$ in $W_0 \cap Y$ or $V$ follows simply because $\nabla\psi = -u^\perp$ is in $W^1(\Omega)$. For $u$ in $V$ it is also true that $\nabla\psi = 0$ on $\Gamma$ so $\psi$ can be taken to lie in $W_0^2(\Omega)$. □

Closely related to Lemma 2.10 is Lemma 2.11, a form of the Biot-Savart law.

**Lemma 2.11.** *The operator $\omega$ is a continuous linear bijection between the following pairs of spaces: $W_0$ and $W^{-1}(\Omega)$, $W_0 \cap X$ and $L^2(\Omega)$, $W_0 \cap X_0^2$ and $W_0^1(\Omega)$.*

*Proof.* That $\omega$ has the domains and ranges stated and that it is continuous follows directly from the definitions of the spaces.

For $\omega$ in $W^{-1}(\Omega)$ let $\psi$ in $W_0^1(\Omega)$ solve $\Delta\psi = \omega$ on $\Omega$ as in Lemma 2.7 and let $u = \nabla^\perp\psi$. Then $\omega(u) = \omega$ and if $\omega(v) = \omega$ as well for $v$ in $W_0$ then $\omega(u - v) = 0$ implying that $u - v$ is in $H_c$. But $u - v$ is also in $W_0$ so $u - v = 0$. Thus, $u = \omega^{-1}(\omega)$ with $\|u\|_H = \|\nabla\psi\|_{L^2} \le C\|\omega\|_{W^{-1}(\Omega)}$ by Lemma 2.7, showing that $\omega^{-1}$ is defined and bounded and hence continuous, since it is clearly linear.

For $\omega$ in in $L^2(\Omega)$ or $W_0^1(\Omega)$ the same argument applies, though now we use either $\|u\|_X = \|\nabla\psi\|_{L^2} + \|\omega(u)\|_{L^2} \le C\|\omega\|_{L^2} + \|\omega\|_{L^2}$ or $\|u\|_{X_0^2} = \|\nabla\psi\|_{L^2} + \|\omega(u)\|_{W^1} \le C\|\omega\|_{L^2} + \|\omega\|_{W^1} \le C\|\omega\|_{W^1}$ to demonstrate the continuity of $\omega^{-1}$. □

**Corollary 2.12.** *$X$ is dense and compactly embedded in $H$ and $X_0^2$ is dense and compactly embedded in $X$.*

*Proof.* Let $A = L^2(\Omega), B = W^{-1}(\Omega)$ or $A = W_0^1(\Omega), B = L^2(\Omega)$. In both cases $A$ is dense and compactly embedded in $B$. Density is transferred to the image spaces $\omega^{-1}(A)$ and $\omega^{-1}(B)$ by virtue of $\omega^{-1}$ being a continuous surjection. The property that the spaces are compactly embedded transfers to the image spaces by virtue of $\omega$ being bounded (since it is continuous linear) along with $\omega^{-1}$ being a continuous surjection. □

We also have the following useful decomposition of $L^2(\Omega)$, variously named after some combination of Leray, Helmholtz, and Weyl:

**Lemma 2.13.** *For any $u$ in $(L^2(\Omega))^2$ there exists a unique $v$ in $H$ and $p$ in $W^1(\Omega)$ such that $u = v + \nabla p$.*

*Proof.* This follows, for instance, from Theorem 1.1 p. 107 of [11], which holds for an arbitrary domain, along with Lemma 2.9. □

The mapping $u \mapsto v$, with $u$ and $v$ as in Lemma 2.13, defines the *Leray* projector $\mathbb{P}$ from $(L^2(\Omega))^2$ onto $H$.

A slight strengthening of Poincare's inequality holds on $Y$ (and so on $V$) when $\Omega$ is simply connected:

**Lemma 2.14.** *For any $u$ in $W_0 \cap X$,*

$$\|u\|_{L^2(\Omega)} \leq C \|\omega(u)\|_{L^2(\Omega)}, \tag{2.3}$$

*and when $\Gamma$ is $C^2$,*

$$\|\nabla u\|_{L^2(\Omega)} \leq C \|\omega(u)\|_{L^2(\Omega)}. \tag{2.4}$$

*Proof.* As in the proof of Lemma 2.10, $u = \nabla^\perp \psi$ for $\psi$ in $W_0^1(\Omega)$ with $\Delta\psi = \omega(u)$ in $L^2(\Omega)$, and $\|\psi\|_{L^2(\Omega)} \leq \|\psi\|_{W^1(\Omega)} \leq C \|\omega(u)\|_{L^2(\Omega)}$ by Lemma 2.7. But $\nabla\psi$ is in $E(\Omega)$ and $\psi$ is in $W^1(\Omega)$ so by Lemma 2.4 we can integrate by parts to give $(\omega(u), \psi) = (\Delta\psi, \psi) = -(\nabla\psi, \nabla\psi) = -\|u\|_{L^2(\Omega)}^2$. Hence by the Cauchy-Schwarz inequality,

$$\|u\|_{L^2(\Omega)}^2 \leq \|\psi\|_{L^2(\Omega)} \|\omega(u)\|_{L^2(\Omega)} \leq C \|\omega(u)\|_{L^2(\Omega)}^2,$$

giving Equation (2.3).

When $\Gamma$ is $C^2$, using Lemma 2.7,

$$\begin{aligned} \|\nabla u\|_{L^2(\Omega)} &= \|\nabla\nabla\psi\|_{L^2(\Omega)} \leq \|\psi\|_{W^2(\Omega)} \leq C \|\Delta\psi\|_{L^2(\Omega)} \\ &= C \|\omega(u)\|_{L^2(\Omega)}, \end{aligned}$$

giving Equation (2.4). □

**Corollary 2.15.** *If $\Gamma$ is $C^2$ and has a finite number of components then any $u$ in $H$ with $\omega(u)$ in $L^2(\Omega)$ is also in $Y$, and*

$$\|\nabla u\|_{L^2(\Omega)} \leq C \left[ \|\omega(u)\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right].$$

*Proof.* This inequality follows from the basic estimate of elliptic regularity theory. □

**Corollary 2.16.** *When $\Gamma$ is $C^2$ and has a finite number of components,*

$$X = Y, \quad X^2 = Y^2 = H \cap W^2(\Omega),$$
$$X_0^2 = Y_0^2 = \left\{u \in H \cap W^2(\Omega) \colon \omega(u) = 0 \text{ on } \Gamma\right\}.$$

*Proof.* The first identity follows from Corollary 2.15 and the second and third from the identity $\Delta u = \nabla^\perp \omega$ and Lemma 2.7. □

We will find a need for the trace operator of Proposition 2.17 in Section 8.

**Proposition 2.17.** *Assume that $\Gamma$ is $C^2$ and has a finite number of components, and let*

$$U = \left\{\omega \in L^2(\Omega) \colon \Delta\omega \in L^2(\Omega)\right\}$$

*endowed with the norm, $\|\omega\|_U = \|\omega\|_{L^2(\Omega)} + \|\Delta\omega\|_{L^2(\Omega)}$. There exists a linear continuous trace operator $\gamma_\omega \colon U \to W^{-1/2}(\Gamma)$ such that $\gamma_\omega\omega$ is the restriction of $\omega$ to $\Gamma$ for all $\omega$ in $C^\infty(\overline{\Omega})$. For any $\alpha$ in $W_0^1(\Omega) \cap W^2(\Omega)$,*

$$(\gamma_\omega\omega, \nabla\alpha \cdot \mathbf{n})_{W^{-1/2}(\Gamma), W^{1/2}(\Gamma)} = (\Delta\alpha, \omega) - (\alpha, \Delta\omega). \tag{2.5}$$

**Lemma 2.18.** *For any $f$ in $L^2(\Omega)$ and $a$ in $(W^{1/2}(\Gamma))^2$ satisfying the compatibility condition,*

$$\int_\Omega f = \int_\Gamma a \cdot \mathbf{n}$$

*there exists a (non-unique) solution $v$ in $W^1(\Omega)$ to $\operatorname{div} v = f$ in $\Omega$, $v = a$ on $\Gamma$.*

*Proof.* This follows from Lemma 3.2 p. 126-127, Remark 3.3 p. 128-129, and Exercise 3.4 p. 131 of [11] (and see the comment on p. 67 of [1]). □

**Lemma 2.19.** *Define $\gamma_{\boldsymbol{\tau}} \colon Y \to L^2(\Gamma)$ by $\gamma_{\boldsymbol{\tau}} v = \gamma_0 v \cdot \boldsymbol{\tau}$ for any $v$ in $Y$. When $\Gamma$ is $C^2$, $\gamma_{\boldsymbol{\tau}}$ maps $Y$ onto $W^{1/2}(\Gamma)$. When $\Gamma$ is $C^2$ and has a finite number of components, $\gamma_{\boldsymbol{\tau}}(W_0 \cap Y)$ is dense in $W^{1/2}(\Gamma)$.*

*Proof.* Assume that $\Gamma$ is $C^2$ and let $g$ lie in $W^{1/2}(\Gamma)$. Then since $\Gamma$ is $C^2$, $g\boldsymbol{\tau}$ is also in $W^{1/2}(\Gamma)$ by Corollary 2.3, and by Lemma 2.18 there exists a vector field $v$ in $W^1(\Omega)$ with $\operatorname{div} v = \int_\Gamma g\boldsymbol{\tau} \cdot \mathbf{n} = 0$ and $v = g\boldsymbol{\tau}$ on $\Gamma$. Thus, in fact, $v$ lies in $Y$, which shows that $\gamma_{\boldsymbol{\tau}}(Y)$ maps onto $W^{1/2}(\Gamma)$. If $\Gamma$ has a finite number of components, then $H_c \cap Y$ is finite-dimensional and so is its image under this map; hence the image of $W_0 \cap Y$ is dense in $W^{1/2}(\Gamma)$. □

***Proof of Proposition 2.17.*** Assume first that $\omega$ is in $C^\infty(\overline{\Omega})$, let $\alpha$ be in $W_0^1(\Omega) \cap W^2(\Omega)$, and let $v = \nabla^\perp\alpha$, so that $v$ lies in $W_0 \cap Y$ with $\Delta\alpha = \omega(v)$.

Then

$$(\alpha, \Delta\omega) = -(\nabla\alpha, \nabla\omega) + \int_\Gamma (\nabla\overline{\omega} \cdot \mathbf{n})\alpha = -(\nabla\alpha, \nabla\omega)$$
$$= (\Delta\alpha, \omega) - \int_\Gamma (\nabla\overline{\alpha} \cdot \mathbf{n})\omega = (\Delta\alpha, \omega) - \int_\Gamma \omega\overline{v} \cdot \boldsymbol{\tau}.$$

From this calculation it follows that for any choice of $v$ (equivalently, by Lemma 2.10, of $\alpha$) with a given value of $\overline{v} \cdot \boldsymbol{\tau}$ on $\Gamma$ the value of $(\Delta\alpha, \omega) - (\alpha, \Delta\omega)$ is the same.

Now, because of Lemma 2.19, we can define $\gamma_\omega(\omega)$ to be that unique element of $W^{-1/2}(\Gamma)$ such that Equation (2.5) holds. This gives a linear mapping from $U$ to $W^{-1/2}(\Gamma)$ whose restriction to $C^\infty(\overline{\Omega})$ is the classical trace.

To establish the continuity of this mapping, let $a$ be any element of $W^{1/2}(\Gamma)$. If $\Omega$ is simply connected then $a = v \cdot \boldsymbol{\tau} = \nabla^\perp\alpha \cdot \boldsymbol{\tau} = \nabla\alpha \cdot \mathbf{n}$ for some $v$ in $Y$ or equivalently for some $\alpha$ in $W_0^1(\Omega) \cap W^2(\Omega)$. Then

$$(\gamma_\omega\omega, a)_{W^{-1/2}(\Gamma), W^{1/2}(\Gamma)} = |(\Delta\alpha, \omega) - (\alpha, \Delta\omega)| \leq C \|\Delta\alpha\|_{L^2(\Omega)} \|\omega\|_U$$
$$\leq C \|\nabla\alpha\|_{W^1(\Omega)} \|\omega\|_U \leq C \|\nabla\alpha\|_{W^{1/2}(\Gamma)} \|\omega\|_U$$
$$= C \|\nabla\alpha \cdot \mathbf{n}\|_{W^{1/2}(\Gamma)} \|\omega\|_U = C \|a\|_{W^{1/2}(\Gamma)} \|\omega\|_U .$$

Here, we Lemma 2.7 in the first and second inequalities and the continuity of the inverse of the standard trace operator in the third inequality. Also, the second-to-last equality holds because $\alpha$ has the constant value of zero on $\Gamma$ so $\nabla\alpha \cdot \boldsymbol{\tau} = 0$ and $|\nabla\alpha| = |\nabla\alpha \cdot \mathbf{n}|$. This shows that the mapping is bounded and hence continuous.

When $\Omega$ is multiply connected the argument is the same except that we must employ a simple density argument using Lemma 2.19. □

## 3. Strong Formulations of three eigenvalue problems

Assume for the moment that $\Gamma$ is $C^2$. Then given any $u$ in $V \cap W^2(\Omega)$, the (classical) Stokes operator $A_S$ applied to $u$ is that unique element $A_S u$ of $H$ such that $\Delta u + A_S u = \nabla p$ for some harmonic pressure field $p$. Equivalently, $A_S = -\mathbb{P}\Delta$, $\mathbb{P}$ being the Leray projector, defined following Lemma 2.13. The operator $A_S$ maps $V \cap W^2(\Omega)$ onto $H$ (see, for instance, p. 49-50 of [7] for more details), is strictly positive-definite, self-adjoint, and as a map from $V$ to $V^*$, the composition of $A_S^{-1}$ with the inclusion map of $V$ into $V^*$ is compact. It follows that $\{u_j\}$ is complete in $H$ (and in $V$) with corresponding eigenvalues $\{\nu_j\}$, $0 < \nu_1 \leq \nu_2 \leq \cdots$, $\nu_j \to \infty$ as $j \to \infty$. Also, the eigenfunctions are orthogonal in both $H$ and $V$.

When $\Gamma$ is only locally Lipschitz, $-\mathbb{P}\Delta$ is only known to be symmetric on $V \cap W^2(\Omega)$, not self-adjoint. Thus, we define $A_S$ to be the Friedrich's extension, as an operator on $H$, of $-\Delta$ defined on $V \cap C_0^\infty(\Omega)$. A concrete description of its domain, $D(A_S)$, in terms of more familiar spaces is not known, though $V \cap H^2(\Omega) \subseteq D(A_S) \subseteq V$. In three dimensions, tighter inclusions have been obtained: see, for instance, [3]. In any case, basic properties of the Friedrich's extension insure that $A_S$ is strictly positive-definite, self-adjoint, and maps $D(A_S)$ bijectively onto $H$.

**Definition 3.1.** A strong eigenfunction $u_j \in V \cap X^2$ of $A_S$ with eigenvalue $\nu_j > 0$ satisfies, for some $p_j$ in $W^1(\Omega)$,

$$\begin{cases} \Delta u_j + \nu_j u_j = \nabla p_j, \quad \Delta p_j = 0, \quad \operatorname{div} u_j = 0 & \text{in } \Omega, \\ u_j = 0 & \text{on } \Gamma. \end{cases} \tag{3.1}$$

Taking the curl of Equation (3.1), we see that the vorticity $\omega_j = \omega(u_j)$ satisfies

$$\begin{cases} \Delta \omega_j + \nu_j \omega_j = 0 & \text{in } \Omega, \\ u_j = 0 & \text{on } \Gamma. \end{cases} \tag{3.2}$$

That is, $\omega_j$ is an eigenfunction of the negative Laplacian, but with boundary conditions on the velocity $u_j$.

Let $\psi_j$ be the stream function for $u_j$ given by Lemma 2.10, so $u_j = \nabla^\perp \psi_j$. Then $\omega_j = \Delta \psi_j$ and $\nabla \psi_j = -u_j^\perp = 0$ on $\Gamma$. Since $\psi_j$ is determined only up to a constant we can then assume that $\psi_j = 0$ on $\Gamma$. Thus, $\psi_j$ satisfies

$$\begin{cases} \Delta\Delta \psi_j + \nu_j \Delta \psi_j = 0 & \text{in } \Omega, \\ \nabla \psi_j \cdot \mathbf{n} = \psi_j = 0 & \text{on } \Gamma. \end{cases} \tag{3.3}$$

This is the eigenvalue problem for the clamped buckling plate (see, for instance, [24, 2]).

Temam exploits the similar correspondence between the Stokes problem and the biharmonic problem in the proof of Proposition I.2.3 of [25] to obtain a relatively simple proof of the regularity of solutions to the Stokes problem in two dimensions with at least $C^2$ regularity of the boundary. Also, as pointed out by Ashbaugh in [2], there is a similar correspondence between the eigenvalue problems for the Dirichlet Laplacian and Equation (3.3) with the boundary condition $\nabla \psi_j \cdot \mathbf{n} = 0$ replaced by $\Delta \psi_j = 0$. This is the correspondence we exploit in the proof of Theorem 1.1, though we view the correspondence as being that given in Lemma 2.11, instead.

What we have shown is that given $u_j$ satisfying Equation (3.1), the corresponding stream function $\psi_j$ satisfies Equation (3.3). Conversely, given $\psi_j$ satisfying Equation (3.3), $\omega_j = \Delta \psi_j$ and $u_j = \nabla^\perp \psi_j$ satisfy Equation (3.2) and one can show, at least for sufficiently smooth boundaries, that $u_j$ satisfies Equation (3.1). Thus, the eigenvalue problems for the Stokes operator and the clamped buckling plate are equivalent.

Returning to Equation (3.1), if we use instead the boundary conditions employed by J.L. Lions in [19] p. 87-98 and P.L. Lions in [20] p. 129-131,

$$u_j \cdot \mathbf{n} = 0, \quad \omega_j = 0 \text{ on } \Gamma, \tag{3.4}$$

which we call *Lions boundary conditions*, we obtain the eigenvalue problem for the Dirichlet Laplacian of Definition 3.2.

**Definition 3.2.** A strong eigenfunction $\omega_j \in W_0^1(\Omega)$ of the Dirichlet Laplacian, $-\Delta_D$, with eigenvalue $\lambda_j > 0$ satisfies

$$\begin{cases} \Delta\omega_j + \lambda_j\omega_j = 0 & \text{in } \Omega, \\ \omega_j = 0 & \text{on } \Gamma. \end{cases} \tag{3.5}$$

Using Lemma 3.4, we can recover the divergence-free velocity $u_j$ in $X_0^2$ uniquely from a vorticity in $W_0^1(\Omega)$ under the constraint that $u_j \cdot \mathbf{n} = 0$, leading to the eigenvalue problem in Definition 3.3 for an operator $A_L$, which we will call the Stokes operator with Lions boundary conditions. (We use $\lambda_j^*$ in place of $\lambda_j$ because of the presence of zero eigenvalues.)

**Definition 3.3.** A strong eigenfunction $u_j \in X_0^2$ of $A_L$ with eigenvalue $\lambda_j^* > 0$ satisfies

$$\begin{cases} \Delta u_j + \lambda_j^* u_j = 0, \quad \operatorname{div} u_j = 0 & \text{in } \Omega, \\ u_j \cdot \mathbf{n} = 0, \quad \omega(u_j) = 0 & \text{on } \Gamma. \end{cases} \tag{3.6}$$

What we have done is to define the eigenvalue problem for the operator $A_L$ before defining the operator itself. In fact, $A_L \colon X_0^2 \to H$ with $A_L u = -\Delta u$. That is, $A_L$ is simply the negative Laplacian on $X_0^2$.

To see that $A_L$ is well-defined, observe that for any $u$ in $X_0^2$, $\Delta u \cdot \mathbf{n} = \nabla^\perp \omega(u) \cdot \mathbf{n} = -\nabla\omega(u) \cdot \boldsymbol{\tau} = 0$, since $\omega(u)$ is constant (namely, zero) along $\Gamma$. (Another way of viewing this is that there is no need for a Leray projector in $X_0^2$, making the Stokes operator on $X_0^2$ akin to the Stokes operator on $H \cap W^2(\Omega)$ for a periodic domain, which of course has no boundary. This is one reason that the use of the boundary conditions of Equation (3.4) in [19] and [20] is so effective.)

**Lemma 3.4.** *Given $\omega$ in $W_0^1(\Omega)$ that satisfies*

$$\begin{cases} \Delta\omega + \lambda\omega = 0 & \textit{in } \Omega, \\ \omega = 0 & \textit{on } \Gamma \end{cases}$$

*with $\lambda > 0$ there exists a unique $u$ in $X_0^2$ such that $\omega = \omega(u)$ and*

$$\begin{cases} \Delta u + \lambda u = 0, \quad \operatorname{div} u = 0 & \textit{in } \Omega, \\ u \cdot \mathbf{n} = 0, \quad \omega(u) = 0 & \textit{on } \Gamma. \end{cases}$$

*Proof.* Let $v = \omega^{-1}(\omega)$, which lies in $W_0 \cap X_0^2$ by Lemma 2.11. Then $\Delta v = \nabla^\perp\omega$ is in $L^2(\Omega)$, so $w = \Delta v + \lambda v$ is a divergence-free vector field in $L^2(\Omega)$. Hence, by Lemma 2.13, $w = h + \nabla p$ for a unique vector field $h$ in $H$ and an harmonic scalar field $p$ in $W^1(\Omega)$ satisfying $\nabla p \cdot \mathbf{n} = w \cdot \mathbf{n} = \Delta v \cdot \mathbf{n}$ on $\Gamma$. (Since $\operatorname{div} \Delta v = 0$, $\Delta v$ is in $E(\Omega)$ so $\Delta v \cdot \mathbf{n}$ is in $W^{-1/2}(\Gamma)$ by Lemma 2.9.)

But $\Delta v \cdot \mathbf{n} = \nabla^\perp\omega(v) \cdot \mathbf{n} = \nabla^\perp\omega \cdot \mathbf{n} = -\nabla\omega \cdot \boldsymbol{\tau} = 0$ on $\Gamma$, where $\omega$ has the constant value of zero. Thus, $\Delta p = 0$ in $\Omega$ with $\nabla p \cdot \mathbf{n} = 0$ on $\Gamma$, so $\nabla p \equiv 0$, and thus $w = h$ and so lies in $H$. Also, $\omega(w) = \Delta\omega(v) + \lambda\omega(v) = \Delta\omega + \lambda\omega = 0$.

Then $u = v - (1/\lambda)w$ is in $H$, and using $\Delta w = \nabla^\perp\omega(w) = 0$ we see that

$$\Delta u + \lambda u = \Delta v + \lambda v - w = w - w = 0,$$

which gives the boundary value problem for $u$ in the statement of the lemma. □

## 4. Weak formulations of the eigenvalue problems

To establish the existence of the eigenfunctions in Section 3 (Proposition 4.10) we work with their weak formulation, then show that these weak formulations are equivalent to those of Section 3 (for $A_S$, though, only when the boundary or the eigenfunctions are sufficiently regular). The formulations for $A_S$ and $A_L$ are modelled along the lines of the formulation in Definition 4.2 for the Dirichlet Laplacian, which is classical (see, for instance, Chapter 1 of [14]).

**Definition 4.1.** The vector field $u_j$ in $V$ is a weak eigenfunction of $A_S$ with eigenvalue $\nu_j > 0$ if

$$(\omega(u_j), \omega(v)) - \nu_j(u_j, v) = 0 \quad \forall v \in V.$$

**Definition 4.2.** The scalar field $\omega_j$ in $W_0^1(\Omega)$ is a weak eigenfunction for the Dirichlet Laplacian with eigenvalue $\lambda_j > 0$ if

$$(\nabla\omega_j, \nabla\alpha) - \lambda_j(\omega_j, \alpha) = 0 \quad \forall \alpha \in W_0^1(\Omega).$$

**Definition 4.3.** The vector field $u_j$ in $W_0 \cap X$ is a weak eigenfunction for $A_L$ for $\lambda_j^* > 0$ if

$$(\omega(u_j), \omega(v)) - \lambda_j^*(u_j, v) = 0 \quad \forall v \in W_0 \cap X. \tag{4.1}$$

Any vector in $H_c$ is an eigenfunction of $A_L$ with zero eigenvalue.

**Proposition 4.4.** *In Definition 4.3 the eigenfunction $u_j$ for $\lambda_j^* > 0$ and the test function $v$ can be taken to lie in $X$.*

*Proof.* Suppose we change Definition 4.3 to assume that $u_j$ and the test function $v$ lie in $X$. Then in particular,

$$(\omega(u_j), \omega(v)) - \lambda_j^*(u_j, v) = -\lambda_j^*(u_j, v) = 0 \text{ for all } v \in H_c.$$

That is, $u_j$ is normal to any vector in $H_c$ and so lies in $W_0 \cap X$. But then knowing that $u_j$ lies in $W_0 \cap X$ it follows that for any $v$ in $H_c$, $(\omega(u_j), \omega(v)) - \lambda_j^*(u_j, v) = 0$; that is, one need only use test functions in $W_0 \cap X$. Thus, the more stringent requirement for being a weak eigenfunction of $A_L$ reduces to the less stringent requirement, meaning that the two are equivalent. □

**Proposition 4.5.** *A strong eigenfunction of $A_S$ is a weak eigenfunction of $A_S$; a weak eigenfunction of $A_S$ lying in $X^2$ is a strong eigenfunction of $A_S$.*

*Proof.* If $u_j$ is a strong eigenfunction of $A_S$ as in Definition 3.1 then applying Corollary A.1, for all $v$ in $V$,

$$(\omega(u_j), \omega(v)) - \nu_j(u_j, v) = -(\Delta u_j + \nu_j u_j, v) = -(\nabla p_j, v) = 0. \tag{4.2}$$

Thus, $u_j$ is a weak eigenfunction of $A_S$ as in Definition 4.1.

Conversely, suppose that $u_j$ is a weak eigenfunction of $A_S$ as in Definition 4.1 for which $\omega(u_j)$ happens to lie in $W^1(\Omega)$. Letting $v$ lie in $V$, $(\omega(u_j), \omega(v)) - \nu_j(u_j, v) = 0$, and we have sufficient regularity of $u_j$ and $v$ to apply Corollary A.1 as above to give $(\Delta u_j + \nu_j u, v) = 0$ for all $v$ in $V$. From Lemma 2.9 we conclude that $\Delta u_j + \nu_j u = \nabla p_j$ for some harmonic pressure field $p_j$ in $W^1(\Omega)$, since $\Delta u_j + \nu_j u$ is in $L^2(\Omega)$. This shows that $u_j$ is a strong eigenfunction of $A_S$ as in Definition 3.1. □

**Proposition 4.6.** *Definitions 3.2 and 4.2 are equivalent as, too, are Definitions 3.3 and 4.3. When $\Gamma$ is $C^2$, Definitions 3.1 and 4.1 are equivalent.*

*Proof.* If $u_j$ is a strong eigenfunction of $A_L$ as in Definition 3.3 then by virtue of Corollary A.1, for all $v$ in $W^1(\Omega)$,

$$\begin{aligned}(\omega(u_j), \omega(v)) - \lambda_j^*(u_j, v) &= -(\Delta u_j, v) + \int_\Gamma \omega(u_j)\overline{v} \cdot \boldsymbol{\tau} - \lambda_j^*(u_j, v) \\ &= -(\Delta u_j + \lambda_j^* u_j, v) = 0.\end{aligned} \tag{4.3}$$

It follows that $u_j$ is a weak eigenfunction of $A_L$ as in Definition 4.3.

Now suppose that $u_j$ is a weak eigenfunction of $A_L$ as in Definition 4.3. Let $\psi_j$ be the stream function for $u_j$ lying in $W_0^1(\Omega)$ given by Lemma 2.10. Then for all $v$ in $X$,

$$\begin{aligned}(u_j, v) = (\nabla^\perp \psi_j, v) = -(\nabla \psi_j, v^\perp) &= (\psi_j, \operatorname{div} v^\perp) - \int_\Gamma (v^\perp \cdot \mathbf{n})\psi_j \\ &= -(\psi_j, \omega(v)).\end{aligned}$$

Hence, by virtue of Proposition 4.4, for all $v$ in $X$,

$$(\omega(u_j) + \lambda_j^* \psi_j, \omega(v)) = (\Delta \psi_j + \lambda_j^* \psi_j, \omega(v)) = 0.$$

Then since by Lemma 2.11 $\omega(v)$ ranges over all of $L^2(\Omega)$, $\Delta \psi_j + \lambda_j^* \psi_j = 0$ so $\omega_j = -\lambda_j^* \psi_j$ lies in $W_0^1(\Omega)$. Thus, $\Delta u_j = \nabla^\perp \omega_j$ is in $L^2(\Omega)$ so $u_j$ is a strong eigenfunction of $A_L$ as in Definition 3.3.

A strong eigenfunction of $A_S$ is a weak eigenfunction of $A_S$ by Proposition 4.5.

Suppose that $u_j$ is a weak eigenfunction of $A_S$ as in Definition 4.1 and that $\Gamma$ is $C^2$. Let $v$ lie in $\mathcal{V}$. Then

$$\begin{aligned}(\omega(u_j), \omega(v)) &= -(\omega(u_j), \operatorname{div} v^\perp) = (\nabla \omega(u_j), v^\perp) = -(\nabla^\perp \omega(u_j), v) \\ &= -(\Delta u_j, v).\end{aligned}$$

Hence,

$$(\Delta u_j + \nu_j u_j, v) = 0 \text{ for all } v \in \mathcal{V}$$

so by Lemma 2.9

$$\Delta u_j + \nu_j u_j = \nabla p_j \tag{4.4}$$

for some $p_j$ in $\mathcal{D}'(\Omega)$.

Now, by Proposition I.2.3 of [25], there exists $w$ in $V \cap W^2(\Omega)$, $q$ in $W^1(\Omega)$ satisfying

$$\Delta w + \nu_j u_j = \nabla q.$$

(This is the only place in which we require $\Gamma$ to be $C^2$.)

Define the bilinear form $a$ on $V \times V$ by $a(u, v) = (\omega(u), \omega(v))$. Then $a(u, v) = (\nabla u, \nabla v)$ by Corollary A.3 so $a(u, u) = \|u\|_V^2$ and we can apply the Lax-Milgram theorem to conclude that $w = u_j$. Hence, $u_j$ is in $V \cap W^2(\Omega)$ showing that it is a strong eigenfunction of $A_S$.

That a strong eigenfunction of $-\Delta_D$ is weak is classical. It is also classical that for a weak eigenfunction, $\omega_j$ is in $C^\infty(\Omega)$, which is enough to conclude that $\Delta \omega_j$ is in $L^2(\Omega)$. □

**Remark 4.7.** When $\Gamma$ is $C^2$, in fact the eigenfunctions of $A_L$ and $A_S$ lie in $W^2(\Omega)$, as can seen by the proof of Proposition 4.6 for $A_L$ and by, for instance, Proposition I.2.3 of [25] for $A_S$.

**Proposition 4.8.** *There exists a bijection between the strong eigenfunctions of $A_L$ having positive eigenvalues and the weak eigenfunctions of the Dirichlet Laplacian, with a corresponding bijection between the eigenvalues.*

*Proof.* By Lemma 2.11 for any $u$ in $W_0 \cap X_0^2$ there exists $\omega(u)$ in $W_0^1(\Omega)$ and this gives a bijection between the spaces. Also by Lemma 2.11 and its proof, for any $v$ in $W_0 \cap X_0^2$ there exists $\omega(v)$ in $W_0^1(\Omega)$, and associated to $v$ is its stream function $\psi$ in $W_0^1(\Omega)$ with $\Delta \psi = \omega(v)$. With $u$, $v$, and $\psi$ as above,

$$\begin{aligned}
\frac{(\nabla \omega, \nabla \psi)}{(\omega, \psi)} &= \frac{-(\omega, \Delta \psi) + \int_\Gamma (\nabla \psi \cdot \mathbf{n}) \omega}{-(\operatorname{div} u^\perp, \psi)} \\
&= \frac{-(\omega(u), \omega(v))}{(u^\perp, \nabla \psi) - \int_\Gamma (u^\perp \cdot \mathbf{n}) \psi} = \frac{-(\omega(u), \omega(v))}{-(u, \nabla^\perp \psi)} \\
&= \frac{(\omega(u), \omega(v))}{(u, v)}.
\end{aligned}$$

We applied Lemma 2.4 twice, the first time using $\omega$ in $W_0^1(\Omega)$ with $\nabla \psi$ in $E(\Omega)$ and the second time using $\psi$ in $W_0^1(\Omega)$ with $u^\perp$ in $E(\Omega)$.

By the bijections mentioned above, this shows that if $\omega$ is a weak eigenfunction of $-\Delta_D$ then $u = \omega^{-1}(\omega)$ is a weak eigenfunction of $A_L$ (also using Corollary 2.12) which lies in $X_0^2$, and hence is a strong eigenfunction of the $A_L$ by Proposition 4.6. The converse follows from the same equality. □

**Corollary 4.9.** *There exists a bijection between the weak eigenfunctions of $A_L$ having positive eigenvalues and the weak eigenfunctions of the Dirichlet Laplacian, with a corresponding bijection between the eigenvalues; that is, $\lambda_k^* = \lambda_k$ for all $k$.*

*Proof.* Combine Propositions 4.6 and 4.8. □

**Proposition 4.10.** *There exists a sequence of weak eigenfunctions for each of our three eigenvalue problems with spectra increasing to infinity as in Equation (1.1) for* $-\Delta_D$ *and* $A_S$ *and with*

$$\sigma_L = \{\lambda_j\}_{j=1}^{\infty}, \ 0 < \lambda_1 < \lambda_2 \leq \cdots .$$

*If* $\Omega$ *is multiply connected,* $\sigma_L$ *will also include 0. The eigenfunctions of* $-\Delta_D$ *form an orthonormal basis of both* $L^2(\Omega)$ *and* $W_0^1(\Omega)$ *while those of* $A_S$ *form an othogonal basis of both* $H$ *and* $V$*. The eigenfunctions of* $A_L$ *lie in* $C^\infty(\Omega) \cap X_0^2$ *and form an orthogonal basis of both* $H$ *and* $X$*. The eigenfunctions of* $-\Delta_D$ *are in* $C^\infty(\Omega) \cap W^2(\Omega)$*.*

*Proof.* To prove the existence of eigenfunctions of $A_S$, let $G$ be the inverse of $A_S$. Let $u, v$ be in $H$. Since $A_S$ is a bijection from $D(A_S)$ onto $H$, there exists $w$ in $D(A_S)$ such that $v = A_S w$, $w = Gv$. Then because $A_S$ is self-adjoint,

$$(Gu, v) = (Gu, A_S w) = (A_S Gu, w) = (u, w) = (u, Gv),$$

showing that $G$ is symmetric and hence, being defined on all of $H$, it is self-adjoint. The above calculation also shows that $(Gu, u) = (A_S Gu, Gu) = \|\nabla Gu\|_{L^2(\Omega)}^2$, which is positive for all nonzero $u$ in $H$.

But $V$ is compactly embedded in $H$ by Lemma 2.6, so $G$, viewed as a map from $H$ to $H$, is compact. Therefore, $G$ is a compact, positive, self-adjoint operator. The spectral theorem thus gives a complete set of eigenfunctions in $H$ and a discrete set of eigenvalues decreasing to zero; applying $G$ to these eigenfunctions and using the reciprocal of the eigenvalues gives the eigenfunctions and eigenvalues of $A_S$ in the usual way.

The results for $-\Delta_D$ are classical and the results for $A_L$ then follow from Corollary 4.9 or they can be proven directly using an argument similar to that above. □

**Remark 4.11.** Because the strong form of the eigenvalue problem for $A_S$, $\Delta u_j + \lambda_j^* u_j = \nabla p_j$, has a nonzero pressure, the classical interior regularity argument for $-\Delta_D$ cannot be made for $A_S$. To obtain further regularity, one must assume a more regular boundary.

## 5. Min-max formulations of the eigenvalue problems

**Proposition 5.1.** *Let*

$$\begin{aligned}
S_k &= \operatorname{span}\{\textit{first } k \textit{ eigenfunctions of } A_S\}, \\
L_k &= \operatorname{span}\{\textit{first } k \textit{ eigenfunctions of } A_L\}, \\
D_k &= \operatorname{span}\{\textit{first } k \textit{ eigenfunctions of } -\Delta_D\},
\end{aligned}$$

*with* $S_0 = L_0 = D_0 = \{0\}$. *Then*

$$\begin{aligned}
\nu_k &= \min \left\{ R_S(u) \colon u \in S_{k-1}^\perp \cap V \setminus \{0\} \right\}, \\
\lambda_k &= \min \left\{ R_D(\omega) \colon \omega \in D_{k-1}^\perp \cap W_0^1(\Omega) \setminus \{0\} \right\} \\
&= \min \left\{ R_L(u) \colon u \in L_{k-1}^\perp \cap W_0 \cap X \setminus \{0\} \right\}, \\
&= \min \left\{ R_L(u) \colon u \in L_{k-1}^\perp \cap W_0 \cap X_0^2 \setminus \{0\} \right\},
\end{aligned}$$

*where the Rayleigh quotients are*

$$R_S(u) = R_L(u) = \frac{\|\omega(u)\|_{L^2(\Omega)}^2}{\|u\|_{L^2(\Omega)}^2}, \quad R_D(\omega) = \frac{\|\nabla \omega\|_{L^2(\Omega)}^2}{\|\omega\|_{L^2(\Omega)}^2}.$$

*Proof.* The form of the Rayleigh coefficient for $\nu_k$ and that in the first two expressions for $\lambda_k$ come from the weak formulations of the eigenvalue problems in Definitions 4.1 through 4.3. The third expression for $\lambda_k$ follows from the bijection in Lemma 2.11 and the observation that if $u$ is any element of $X_0^2$ then $R_L(u) = R_D(\omega(u))$, as in the proof of Proposition 4.8. □

Define the four functions mapping $\mathbb{R}$ to $\mathbb{Z}$,

$$\begin{aligned}
N_S(\lambda) &= \# \{ j \in \mathbb{N} \colon \nu_j < \lambda \}, \quad & N_L(\lambda) &= \# \{ j \in \mathbb{N} \colon \lambda_j < \lambda \}, \\
\overline{N}_S(\lambda) &= \# \{ j \in \mathbb{N} \colon \nu_j \le \lambda \}, \quad & \overline{N}_L(\lambda) &= \# \{ j \in \mathbb{N} \colon \lambda_j \le \lambda \}.
\end{aligned}$$

Corollary 5.2 follows immediately from Proposition 5.1.

**Corollary 5.2.** *We have,*

$$\begin{aligned}
\overline{N}_S(\lambda) &= \max_{Z \subseteq V} \{ \dim Z \colon R_S(u) \le \lambda \text{ for all } u \in Z \}, \\
\overline{N}_L(\lambda) &= \max_{Z \subseteq W_0 \cap X_0^2} \{ \dim Z \colon R_L(u) \le \lambda \text{ for all } u \in Z \} \\
&= \max_{Z \subseteq W_0 \cap X} \{ \dim Z \colon R_L(u) \le \lambda \text{ for all } u \in Z \}.
\end{aligned}$$

**Remark 5.3.** By Corollary A.3, $R_S(u) = \|\nabla u\|_{L^2(\Omega)}^2 / \|u\|_{L^2(\Omega)}^2$, so $\lambda_k \le \nu_k$ follows from Corollary 5.2. Strict inequality, however, is not so immediate.

## 6. Proof of Theorem 1.1

Lemma 6.1 is the analog of the (only) lemma in [6] and, in fact, follows from it. For completeness we give the full proof.

**Lemma 6.1.** *For all* $\lambda$ *in* $\mathbb{R}$,

$$V \cap \ker \{A_L - \lambda\} \cap X_0^2 = \{0\}.$$

*Proof.* Let $u$ be in $V \cap \ker \{A_L - \lambda\} \cap X_0^2 = \ker \{A_S - \lambda\} \cap X_0^2$, where we used Proposition 4.5. Then

$$\begin{cases} \Delta u + \lambda u = \nabla p, \quad \operatorname{div} u = 0, \quad \Delta \omega + \lambda \omega = 0 & \text{in } \Omega, \\ u = 0, \quad \omega = 0 & \text{on } \Gamma. \end{cases}$$

Because $\omega = 0$ on $\Gamma$, $\nabla p = 0$ on $\Omega$ by Lemma 3.4. Hence, $\nabla \omega = -(\Delta u)^\perp = \lambda u^\perp = 0$ on $\Gamma$. Thus, $\omega$ extended by 0 to all of $\mathbb{R}^2$ lies in $W^1(\mathbb{R}^2)$. Then for all $\psi$ in $\mathcal{S}(\mathbb{R}^2)$,

$$\begin{aligned}(-\Delta\omega, \psi)_{\mathcal{S}'(\mathbb{R}^2),\mathcal{S}(\mathbb{R}^2)} &= (\nabla\omega, \nabla\psi)_{\mathcal{S}'(\mathbb{R}^2),\mathcal{S}(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \nabla\omega \cdot \nabla\overline{\psi} \\ &= \int_\Omega \nabla\omega \cdot \nabla\overline{\psi} = -\int_\Omega \Delta\omega\overline{\psi} + \int_\Gamma (\nabla\omega \cdot \mathbf{n})\overline{\psi} \\ &= \lambda \int_\Omega \omega\overline{\psi} = \lambda \int_{\mathbb{R}^2} \omega\overline{\psi} = (\lambda\omega, \psi)_{\mathcal{S}'(\mathbb{R}^2),\mathcal{S}(\mathbb{R}^2)},\end{aligned}$$

which shows that $\Delta\omega = -\lambda\omega$ as distributions. But $\omega$ is in $W^1(\mathbb{R}^2)$ so, in fact, $\Delta\omega$ is in $W^1(\mathbb{R}^2)$ and $\Delta\omega + \lambda\omega = 0$ on $\mathbb{R}^2$. Moreover, $\omega$ vanishes outside of $\Omega$. But the Laplacian is hypo-elliptic so $\omega$ is real analytic and therefore vanishes on all of $\mathbb{R}^2$.

Now, were $\Omega$ simply connected it would follow immediately that $u \equiv 0$. In any case, observe that $\omega \equiv 0 \implies \Delta u = \nabla^\perp \omega \equiv 0$. But $\Delta u = -\lambda u$ so $u \equiv 0$. □

***Proof of Theorem 1.1***. Let $\lambda > 0$ and choose a subspace $F$ of $V$ of dimension $\overline{N}_S(\lambda)$ with

$$\|\omega(u)\|^2_{L^2(\Omega)} \le \lambda \|u\|^2_{L^2(\Omega)} \text{ for all } u \in F. \tag{6.1}$$

This is possible by the variational formulation of the eigenvalue problem for $A_S$ in Corollary 5.2. By Lemma 6.1,

$$G = F \oplus \left(\ker\{A_L - \lambda\} \cap X_0^2\right)$$

is a direct sum and so has dimension $\overline{N}_S(\lambda) + \dim\ker\{-\Delta_D - \lambda\}$, where we used Propositions 4.5 and 4.8. (Either of the vector spaces above could contain only 0.)

For any $u \in F$, $v \in \ker\{A_L - \lambda\} \cap X_0^2$,

$$\begin{aligned}\|\omega(u+v)\|^2_{L^2(\Omega)} &= \|\omega(u)\|^2_{L^2(\Omega)} + \|\omega(v)\|^2_{L^2(\Omega)} + 2\operatorname{Re}(\omega(u), \omega(v)) \\ &= \|\omega(u)\|^2_{L^2(\Omega)} + \|\omega(v)\|^2_{L^2(\Omega)} + 2\lambda\operatorname{Re}(u, v),\end{aligned}$$

because $(\omega(u), \omega(v)) = \lambda(u, v)$ by Definition 4.3.

Also by Definition 4.3,

$$\|\omega(v)\|^2_{L^2(\Omega)} = \lambda \|v\|^2_{L^2(\Omega)},$$

and combined with Equation (6.1) this gives

$$\begin{aligned}\|\omega(u+v)\|^2_{L^2(\Omega)} &\le \lambda \|u\|^2_{L^2(\Omega)} + \lambda \|v\|^2_{L^2(\Omega)} + 2\lambda\operatorname{Re}(u, v) \\ &= \lambda \|u+v\|^2_{L^2(\Omega)}.\end{aligned}$$

Then by the variational formulation of the eigenvalue problem for $A_L$ in Corollary 5.2 it follows that

$$\overline{N}_L(\lambda) \ge \dim G = \overline{N}_S(\lambda) + \dim\ker\{-\Delta_D - \lambda\}$$

so

$$N_L(\lambda) = \overline{N}_L(\lambda) - \dim\ker\{-\Delta_D - \lambda\} \geq \overline{N}_S(\lambda).$$

Setting $\lambda = \nu_k$ gives

$$N_L(\nu_k) \geq \overline{N}_S(\nu_k) \geq k.$$

In other words, there are at least $k$ eigenvalues in $\sigma_D$ (counted according to multiplicity) strictly less than $\nu_k$; that is, $\lambda_k < \nu_k$. □

## 7. Toward the inequality $\lambda_{k+1} \leq \nu_k$

**Theorem 7.1.** *For each $k$ in $\mathbb{N}$, define $U_R^k = (\nu_k, x)$, where $x$ is the smallest element of $(\sigma_S \cup \sigma_D) \cap (\nu_k, \infty)$, and define $U_L^k = (y, \lambda_k)$, where $y$ is the largest element of $(\sigma_S \cup \sigma_D) \cap (-\infty, \lambda_k)$. (Let $y = -\infty$ if $k = 1$.) Suppose that for some $\lambda$ in $U_R^k$ there exists a nonzero vector field $w$ in $X^2$ and a scalar field $q$ in $W^1(\Omega)$ satisfying the underdetermined problem,*

$$\begin{cases} \Delta w + \lambda w = \nabla q, \quad \operatorname{div} w = 0 & \text{on } \Omega, \\ w \cdot \mathbf{n} = 0 & \text{on } \Gamma, \end{cases} \tag{7.1}$$

*but with the constraint*

$$\int_\Gamma \omega(w)\overline{w} \cdot \boldsymbol{\tau} = \|\omega(w)\|^2_{L^2(\Omega)} - \lambda \|w\|^2_{L^2(\Omega)} \leq 0. \tag{7.2}$$

*Then $\lambda_{k+1} \leq \nu_k$. If for each $k$ there exist $\lambda$ in $U_L^k$, a nonzero vector field $w$ in $X^2$, and a scalar field $q$ in $W^1(\Omega)$ satisfying Equation (7.1) and Equation (7.2) then $\lambda_{k+1} \leq \nu_k$ for all $k$.*

*Proof.* Observe first that $\int_\Gamma \omega(w)\overline{w} \cdot \boldsymbol{\tau} = \|\omega(w)\|^2_{L^2(\Omega)} - \lambda \|w\|^2_{L^2(\Omega)}$ follows from Corollary A.1.

Assume that $\lambda$ in $U_R^k$ and $w$ and $q$ are as in Equation (7.1) and Equation (7.2). Let the set $F$ be defined as in the proof of Lemma 6.1, but let

$$G = F \oplus \operatorname{span}\{w\}.$$

This is a direct sum since otherwise $w$ would be in $\operatorname{span} F$, meaning that it would vanish on $\Gamma$ and so would actually be an eigenfunction of $A_S$; but this is impossible since $\lambda$ is not in $\sigma_S$ by assumption. The dimension of $G$ is $\overline{N}_S(\lambda) + 1$.

Then for any $u$ in $F$ and $c$ in $\mathbb{C}$,

$$\|\omega(u + cw)\|^2_{L^2} = \|\omega(u)\|^2_{L^2} + \|\omega(cw)\|^2_{L^2} + 2\operatorname{Re}(\omega(u), \omega(cw)).$$

But by Corollary A.1,

$$(\omega(u), \omega(w)) = -(\Delta w, u) = (\lambda w, u) - (\nabla q, u) = \lambda(u, w)$$

and

$$\|\omega(w)\|^2_{L^2} \leq \lambda \|w\|^2_{L^2}$$

by Equation (7.2). Also, $\|\omega(u)\|_{L^2}^2 \le \lambda \|u\|_{L^2}^2$, so we can conclude that

$$\begin{aligned}\|\omega(u+cw)\|_{L^2}^2 &\le \lambda \|u\|_{L^2}^2 + \lambda \|cw\|_{L^2}^2 + 2\lambda \operatorname{Re}(u, cw) \\ &= \lambda \|u+cw\|_{L^2}^2 .\end{aligned}$$

Then by the variational formulation of the eigenvalue problem for $A_L$ in Corollary 5.2 it follows that

$$\overline{N}_L(\lambda) \ge \dim G = \overline{N}_S(\lambda) + 1.$$

Because $\lambda$ is larger than $\nu_k$ but smaller than any eigenvalue in $(\sigma_D \cup \sigma_S) \cap (\lambda, \infty)$, $N_L(\lambda) = \overline{N}_L(\nu_k)$ and $\overline{N}_S(\lambda) = \overline{N}_S(\nu_k)$ so

$$\overline{N}_L(\nu_k) \ge \overline{N}_S(\nu_k) + 1 \ge k + 1.$$

In other words, there are at least $k+1$ eigenvalues in $\sigma_D$ (counted according to multiplicity) less-than-or-equal-to $\nu_k$; that is, $\lambda_{k+1} \le \nu_k$. This establishes the result for $\lambda$ in $U_R^k$.

Now assume that for all $k$ there exists a $\lambda$ in $U_L^k$ with $w$ and $q$ as in Equation (7.1) and Equation (7.2). Given $j$ in $\mathbb{N}$, let $\delta$ be the lowest eigenvalue greater than $\nu_j$ in $\sigma_S \cup \sigma_D$. If $\delta$ is in $\sigma_S$, then $\delta = \nu_n$ for some $n > j$, and if $\lambda_{n+1} \le \nu_n$ then it will follow that $\lambda_{j+1} \le \nu_j$ since there are no eigenvalues in $\sigma_D$ between $\nu_j$ and $\nu_n$ (though $\nu_j$, $\nu_n$, or both might also be in $\sigma_D$). We can continue this line of reasoning until eventually we reach a value of $j$ such that the next lowest eigenvalue $\delta$ in $\sigma_S \cup \sigma_D$ is in $\sigma_D$ ($\delta$ might also be in $\sigma_S$, but this will not affect our argument). Then $\delta = \lambda_n$ for some $n$ in $\mathbb{N}$.

Then by assumption there is some $\lambda$ in $U_L^n$ with $w$ and $q$ as in Equation (7.1) and Equation (7.2). But this $\lambda$ is also in $U_R^j$, so we conclude that $\lambda_{j+1} \le \nu_j$, and from our argument above, this inequality holds, then, for all $j$ in $\mathbb{N}$. □

**Remark 7.2.** For $\lambda$ in $\sigma_D$, even if a $w$ exists satisfying the conditions in Equation (7.1) and Equation (7.2), $w$ might be an eigenfunction of $A_L$ and so lie in $\ker\{A_L - \lambda\}$. This means that we cannot extend the argument along the lines in the proof of Theorem 1.1, since $\operatorname{span}\{w\}$ might not be linearly independent of the set $G$ in the proof of that theorem. This prevents us from concluding that $\lambda_{k+1} < \nu_k$ for all $k$, which is in any case not true in general.

The difficulty with applying Theorem 7.1 is that it is relatively easy to find vector fields $w$ satisfying the given conditions in a left neighborhood of $\nu_k$, or perhaps in a right neighborhood of $\lambda_k$, but hard to find ones in the required neighborhoods. We give an example in Section 8.

## 8. Proof of Theorem 1.2 and related issues

Navier slip boundary conditions for the Stokes operator provide a physically justifiable alternative to the classical no-slip boundary conditions used to define $A_S$. To the extent possible, we will work with these boundary conditions with a locally Lipschitz boundary, but we will find that they are really only of use when the boundary is $C^2$ and has a finite number of components. (Observe that under this assumption, by Corollary 2.16, the distinctions we have been making between the "X" spaces and the "Y" spaces disappear.)

To define Navier boundary conditions in the classical sense, we must assume that $\Gamma$ is $C^2$. (Here, as elsewhere in this paper, $C^{1,1}$ would suffice, but introduces added complexities we wish to avoid.) The Navier conditions can be written in the form

$$\omega(u) = (2\kappa - \alpha) u \cdot \boldsymbol{\tau} \text{ on } \Gamma, \tag{8.1}$$

where $\kappa$ is the curvature of the boundary and $\alpha$ is any function in $L^\infty(\Gamma)$.

If $u$ in $H \cap W^2(\Omega)$ satisfies Equation (8.1) then by Corollary A.1, for any $v$ in $X$,

$$(-\Delta u, v) = (\omega(u), \omega(v)) - \int_\Gamma (2\kappa - \alpha) u \cdot \overline{v}.$$

Let

$$H_V = \left\{ u \in H \cap W^2(\Omega) \colon \omega(u) = (2\kappa - \alpha) u \cdot \boldsymbol{\tau} \text{ on } \Gamma \right\},$$

endowed with the same norm as $Y$. We define the operator $A_V \colon Y \to H$ by requiring that

$$\begin{aligned} (A_V u, v) &= (\omega(u), \omega(v)) + \int_\Gamma (\alpha - 2\kappa) u \cdot \overline{v} \\ &= (\nabla u, \nabla v) + \int_\Gamma (\alpha - \kappa) u \cdot \overline{v}, \end{aligned} \tag{8.2}$$

for all $v$ in $Y$. The second equality (which gives the form of the operator $A$ defined on p. 218 of [17]) follows from Lemma A.2, Lemma A.4, and the density of $(C^1(\Omega))^2$ in $Y$.

Now assume that $\Omega$ is bounded and $\Gamma$ is locally Lipschitz. Then the curvature is no longer defined, so we replace the function $\alpha - 2\kappa$ with a function $f$ lying in $L^\infty(\Gamma)$, though we lose in this way the physical meaning. In place of Equation (8.1), we have

$$\omega(u) + f u \cdot \boldsymbol{\tau} = 0 \text{ on } \Gamma \tag{8.3}$$

and

$$(A_V u, v) = (\omega(u), \omega(v)) + \int_\Gamma f u \cdot \overline{v}. \tag{8.4}$$

Observe that the second expression for $A_V$ in Equation (8.2) now has insufficient regularity so it no longer applies.

We define the strong and weak formulation of the eigenvalue problem for $A_V$ as follows:

**Definition 8.1.** A vector field $u_j \in X^2$ is a strong eigenfunction of $A_V$ with eigenvalue $\gamma_j$ if

$$\begin{cases} \Delta u_j + \gamma_j u_j = \nabla p_j, \quad \Delta p_j = 0, \quad \operatorname{div} u_j = 0 & \text{in } \Omega, \\ u_j \cdot \mathbf{n} = 0, \quad \omega(u_j) + f u_j \cdot \boldsymbol{\tau} = 0 & \text{on } \Gamma. \end{cases}$$

**Definition 8.2.** The vector field $u_j$ in $X$ is a weak eigenfunction of $A_V$ with eigenvalue $\gamma_j$ if

$$(\omega(u_j),\, \omega(v)) + \int_\Gamma f u_j \cdot \overline{v} - \gamma_j(u_j, v) = 0 \quad \forall v \in X.$$

**Proposition 8.3.** *If $u_j$ is a strong eigenfunction of $A_V$ then it is a weak eigenfunction of $A_V$; if $u_j$ is a weak eigenfunction of $A_V$ that happens to be in $X^2$ and satisfy $\omega(u_j) + f u_j \cdot \boldsymbol{\tau} = 0$ on $\Gamma$ then $u_j$ is a strong eigenfunction of $A_V$.*

*Proof.* Strong implies weak follows by the integration by parts performed above. For the reverse implication, assume that $u_j$ is a weak eigenfunction of $A_V$ lying in $X^2$. Then choosing $v$ to lie in $V$ it follows that

$$(\omega(u_j),\, \omega(v)) - \gamma_j(u_j, v) = 0 \quad \forall v \in V.$$

Applying Corollary A.1 gives

$$(\Delta u_j + \gamma_j u_j, v) = 0 \quad \forall v \in V,$$

and we conclude that $\Delta u_j + \gamma_j u_j = \nabla p_j$ for some harmonic field $p$ in $W^1(\Omega)$ by Lemma 2.9. $\square$

When $\Gamma$ is $C^2$ and has a finite number of components, we can consider the special case $\alpha = \kappa$, which gives $\omega(u_j) = \kappa u_j \cdot \boldsymbol{\tau}$. It follows from Lemma A.5 that $\nabla u_j \mathbf{n} \cdot \overline{v} = 0$ for any $v$ in $X$. More simply, we can write this as $\nabla u_j \mathbf{n} \cdot \boldsymbol{\tau} = 0$. These boundary conditions imply that for all $v$ in $X$, $(-\Delta u_j, v) = (\nabla u_j, \nabla v)$ (or we can take advantage of the second form of $(A_V u, u)$ in Equation (8.2)), and we can explicitly define such eigenfunctions as follows, though we need no longer assume that the boundary is $C^2$:

**Definition 8.4.** A vector field $u_j \in X^2$ is a strong eigenfunction of $A_N$ if

$$\begin{cases} \Delta u_j + \beta_j u_j = \nabla p_j, \quad \Delta p_j = 0, \quad \operatorname{div} u_j = 0 & \text{in } \Omega, \\ u_j \cdot \mathbf{n} = 0,\ \nabla u_j \mathbf{n} \cdot \boldsymbol{\tau} = 0 & \text{on } \Gamma. \end{cases}$$

**Definition 8.5.** A vector field $u_j$ in $X$ is a weak eigenfunction of $A_N$ if

$$(\nabla u_j, \nabla v) - \beta_j(u_j, v) = 0 \quad \forall v \in X.$$

We also have the following min-max formulations for the eigenvalues of $A_V$ and the special case of $A_N$.

**Proposition 8.6.** *Let*

$$V_k = \operatorname{span}\{\textit{first } k \textit{ eigenfunctions of } A_V\},$$
$$N_k = \operatorname{span}\{\textit{first } k \textit{ eigenfunctions of } A_N\},$$

*with $V_0 = N_0 = \{0\}$. Then*

$$\gamma_k = \min\left\{R_V(u) \colon u \in V_{k-1}^\perp \cap X \setminus \{0\}\right\},$$
$$\beta_k = \min\left\{R_N(u) \colon u \in N_{k-1}^\perp \cap X \setminus \{0\}\right\},$$

*where*

$$R_V(u) = \frac{\|\omega(u)\|^2_{L^2(\Omega)} + \int_\Gamma f\,|u|^2}{\|u\|^2_{L^2(\Omega)}}, \quad R_N(u) = \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}}.$$

*The eigenvalues are real with* $0 = \beta_1 \le \beta_2 \le \cdots$ *and, when* $f$ *is nonnegative,* $0 < \gamma_1 \le \gamma_2 \le \cdots$ *with* $\gamma_k \to \infty$.

*Proof.* Define the operator $T\colon X \to X$ by

$$T = (iI + A_V)^{-1} \circ j,$$

where $I$ is the identity map, $j$ is the inclusion map from $X$ to $X^*$ (which is compact by Corollary 2.12), and $i = \sqrt{-1}$. Then since $(iI + A_V)^{-1}$ is bounded (its norm can be no greater than 1) $T$ is compact, and the spectral theorem provides us with eigenvalues of $T$ accumulating at zero. To each eigenvalue $\lambda$ of $T$ there corresponds an eigenvalue $\gamma = \mu^{-1} - i$ of $A_V$. But $A_V$ is self-adjoint, so $\gamma$ is real. And when $f$ is nonnegative, since $R_V(u)$ is nonnegative, $0 < \gamma_1 \le \gamma_2 \le \cdots$ with $\gamma_k \to \infty$. □

Define the two functions mapping $\mathbb{R}$ to $\mathbb{Z}$,

$$\overline{N}_V(\lambda) = \#\left\{j \in \mathbb{N}\colon \gamma_j \le \lambda\right\}, \quad \overline{N}_N(\lambda) = \#\left\{j \in \mathbb{N}\colon \beta_j \le \lambda\right\}.$$

Corollary 8.7 follows immediately from Proposition 8.6.

**Corollary 8.7.** *We have,*

$$\overline{N}_V(\lambda) = \max_{Z \subseteq X}\left\{\dim Z\colon R_V(u) \le \lambda \text{ for all } u \in Z\right\},$$
$$\overline{N}_N(\lambda) = \max_{Z \subseteq X}\left\{\dim Z\colon R_N(u) \le \lambda \text{ for all } u \in Z\right\}.$$

**Proposition 8.8.** *Assume that* $\Gamma$ *is* $C^2$ *and has a finite number of components and that*

$$f \in C^{1/2+\epsilon}(\Gamma) + W^{1/2+\epsilon}(\Gamma). \tag{8.5}$$

*A weak eigenfunction of* $A_V$ *is a strong eigenfunction of* $A_V$. *In particular, a weak eigenfunction* $u_j$ *of* $A_V$ *satisfies* $\omega(u_j) + f u_j \cdot \boldsymbol{\tau} = 0$ *on* $\Gamma$.

*Proof.* Suppose that $u$ is a weak eigenfunction of $A_V$ as in Definition 8.2 with $\omega = \omega(u)$. Then for any $v$ in $\mathcal{V}$ integration by parts gives $(\Delta u + \lambda u, v) = 0$ so $\Delta u + \lambda u = \nabla p$ by Lemma 2.9, equality holding in terms of distributions. Taking the curl it follows that $\Delta\omega = -\lambda\omega$ so $\omega$ is in $U$ of Proposition 2.17, since $\omega$ is in $L^2$. Thus, by Proposition 2.17, $\omega$ is well-defined on $\Gamma$ as an element of $W^{-1/2}(\Gamma)$.

Let $v$ be any vector in $W_0 \cap Y$ and let $\alpha$ be its associated stream function lying in $W^1_0(\Omega) \cap W^2(\Omega)$ given by Lemma 2.10, so that $\Delta\alpha = \omega(v)$ is in $L^2(\Omega)$. Thus, again by Proposition 2.17, since $\nabla\alpha \cdot \mathbf{n} = -v \cdot \boldsymbol{\tau}$,

$$\begin{aligned}(\gamma_\omega \omega, v \cdot \boldsymbol{\tau})_{W^{-1/2}(\Gamma), W^{1/2}(\Gamma)} &= (\alpha, \Delta\omega) - (\omega(v), \omega)\\ &= -\lambda(\alpha, \omega) - (\omega(v), \omega) = \lambda(u, v) - (\omega(v), \omega).\end{aligned}$$

Here we used,

$$(\alpha, \omega) = -(\alpha, \operatorname{div} u^\perp) = (\nabla\alpha, u^\perp) + \int_\Gamma (u^\perp \cdot \mathbf{n})\alpha = -(v, u),$$

noting that we had enough regularity to apply Corollary A.1.

But because $u$ is a weak eigenfunction of $A_V$, also

$$(fu \cdot \boldsymbol{\tau}, v \cdot \boldsymbol{\tau})_{W^{-1/2}(\Gamma), W^{1/2}(\Gamma)} = \lambda(u, v) - (\omega(v), \omega).$$

Thus, the two boundary integrals are equal, and because of Lemma 2.19, we can conclude that $\omega = -fu \cdot \boldsymbol{\tau}$ on $\Gamma$, and in particular that $\omega$ is in $W^{1/2}(\Gamma)$. (By Corollaries 2.2 and 2.3 and Equation (8.5) we know that $fu \cdot \boldsymbol{\tau}$ is in $W^{1/2}(\Gamma)$.) From this gain of regularity on the boundary, along with $\Delta\omega = -\lambda\omega \in L^2(\Omega)$, we conclude that $\omega$ is in $W^1(\Omega)$, from which it follows that $u$ is a strong solution to $A_V$ as in Definition 8.1.

(The origin of this proof was the proof of Lemma 2.2 of [4].) □

We have the following simple extension of Lemma 6.1:

**Lemma 8.9.** *When $\Gamma$ is $C^2$ and has a finite number of components, and Equation (8.5) holds, for all $\lambda$ in $\mathbb{R}$,*

$$V \cap \ker\{A_V - \lambda\} = \{0\}.$$

*Proof.* By Proposition 8.8, $u$ is a strong eigenfunction of $A_V$ and so satisfies $\omega(u) = -fu \cdot \boldsymbol{\tau} = 0$ on $\Gamma$, and so is a strong eigenfunction of $A_L$. But then $u = 0$ by Lemma 6.1. □

Restricting our attention to the case where $f$ is nonnegative and constant on $\Gamma$ (in which case Equation (8.5) holds), we can write the boundary conditions in Definition 8.1 as $(1-\theta)\omega(u_j) + \theta u_j \cdot \boldsymbol{\tau} = 0$ on $\Gamma$, where $\theta$ lies in $[0, 1]$. When $\theta = 0$, we have the special case of Lions boundary conditions and when $\theta = 1$ we have Dirichlet boundary conditions on the velocity. In Definition 8.2, $f = \theta/(1-\theta)$ for $\theta$ in $[0, 1)$. With this parameterization, we can view $\gamma_j$ as a function of $\theta$. That is, $\gamma_j(\theta)$ is the $j$-th eigenvalue of $A_V$ (or $A_L$ or $A_S$) so, for instance, to each eigenvalue $\gamma_j(\theta)$ of multiplicity $k$ there will be exactly $k$ values of $n$ for which $\gamma_n(\theta) = \gamma_j(\theta)$.

Because $f$ is constant on $\Gamma$, $f$ is certainly in $C^1(\Gamma)$, which is a requirement of Proposition 8.8.

**Proposition 8.10.** *Assume that $\Gamma$ is $C^2$ and has a finite number of components. For all $j$ in $\mathbb{N}$, $\gamma_j \colon [0, 1) \to [\lambda_j, \nu_j)$ and is strictly increasing and continuous.*

*Proof.* To show that $\gamma_j(\theta) < \nu_j$ for $\theta$ in $[0, 1)$ we repeat the proof of Theorem 1.1 using $G = F \oplus \ker\{A_V - \lambda\}$ in place of $F \oplus \ker(\{A_L - \lambda\} \cap X_0^2)$. Let $u \in F$, $v \in \ker\{A_V - \lambda\}$. Then because $v$ is a weak eigenfunction of $A_V$ as in Definition 8.2 and $u$ is zero on the boundary, letting $z = f = \theta/(1-\theta)$, we have

$$(\omega(u), \omega(v)) = \lambda(u, v) - z \int_\Gamma v \cdot \overline{u} = \lambda(u, v).$$

Thus,

$$\begin{aligned}\|\omega(u+v)\|^2_{L^2(\Omega)} &= \|\omega(u)\|^2_{L^2(\Omega)} + \|\omega(v)\|^2_{L^2(\Omega)} + 2\operatorname{Re}(\omega(u), \omega(v)) \\ &= \|\omega(u)\|^2_{L^2(\Omega)} + \|\omega(v)\|^2_{L^2(\Omega)} + 2\lambda \operatorname{Re}(u, v),\end{aligned}$$

as was the case for $A_L$. Now, however,

$$\|\omega(v)\|^2_{L^2(\Omega)} = \lambda \|v\|^2_{L^2(\Omega)} - z \int_\Gamma |v|^2 = \lambda \|v\|^2_{L^2(\Omega)} - z \int_\Gamma |u+v|^2,$$

and combined with Equation (6.1) this gives

$$\begin{aligned}\|\omega(u+v)\|^2_{L^2(\Omega)} &\le \lambda \|u\|^2_{L^2(\Omega)} + \lambda \|v\|^2_{L^2(\Omega)} + 2\lambda \operatorname{Re}(u, v) - z \int_\Gamma |u+v|^2 \\ &= \lambda \|u+v\|^2_{L^2(\Omega)} - z \int_\Gamma |u+v|^2.\end{aligned}$$

Thus, $R_V(u+v) \le \lambda$, and the proof of $\gamma_j(\theta) < \nu_j$ is completed as in the proof of Theorem 1.1.

The argument that $\gamma_j$ is strictly increasing on $[0, 1)$ is more direct, because the variational formulations in Corollary 8.7 for different values of $\theta$ all involve maximums over subspaces of the same space $Y$. (That $\gamma_j$ is nondecreasing on $[0, 1)$ follows immediately from the principle of monotonicity, as in Theorem 2.5.1 p. 21 of [26].)

For $\theta$ in $[0, 1)$, write $A^\theta_V$ for the operator $A_V$ and similarly for $R^\theta_V$ and $\overline{N}^\theta_V$. In particular, $A_L = A^0_V$. Let $f(\theta) = \theta/(1-\theta)$, which we note is an increasing function of $\theta$ on $[0, 1)$.

Now suppose that $\theta, \theta'$ are in $[0, 1)$ with $\theta < \theta'$. Let $\lambda > 0$ and choose a subspace $F$ of $Y$ of dimension $\overline{N}^{\theta'}_V(\lambda)$ with $R^{\theta'}_V \le \lambda$; that is,

$$\|\omega(u)\|^2_{L^2(\Omega)} + \int_\Gamma f(\theta') |u|^2 \le \lambda \|u\|^2_{L^2(\Omega)} \text{ for all } u \in F, \tag{8.6}$$

which is possible by Corollary 8.7. Let

$$G = F \oplus \ker\{A^\theta_V - \lambda\}.$$

This is, in fact, a direct sum, since if a nonzero $u$ lies in both $F$ and $\ker\{A^\theta_V - \lambda\}$ then from Equation (8.6) and Definition 8.2 it follows that

$$\int_\Gamma (f(\theta') - f(\theta)) |u|^2 \le 0.$$

But $f(\theta') - f(\theta)$ is a positive constant on $\Gamma$ so, in fact, $u = 0$ on $\Gamma$ and hence lies in $V$. It follows from Lemma 8.9 that $u$ is identically zero.

This shows that when $\lambda = \gamma_j(\theta)$, $G$ has at least one more element than $F$. But then setting $Z = F$ in the definition of $\overline{N}_V^\theta(\gamma_j(\theta))$ in Corollary 8.7, because $R_V^\theta \le R_V^{\theta'}$, we see that

$$\overline{N}_V^\theta(\gamma_j(\theta)) \ge \dim G > \dim F = \overline{N}_V^{\theta'}(\gamma_j(\theta)),$$

which means that $\gamma_j(\theta) < \gamma_j(\theta')$.

This shows that $\gamma_j$ is strictly increasing. To show that it is continuous, fix $\theta$ in $[0, 1)$ and let $Z$ be any subspace of $Y$ that achieves the maximum in the expression for $k = \overline{N}_V^\theta(\gamma_k(\theta))$ in Corollary 8.7. Here we assume that if $\lambda_k$ is a multiple eigenvalue $k$ is the largest such index.

Choose a basis $(v_1, \dots, v_k)$ for $Z$ and observe that because $R_V(u) = R_V(cu)$ for any nonzero constant $c$,

$$\sup_{u \in Z} R_V^{\theta'}(u) = \max_{u \in Z'} R_V^{\theta'}(u)$$

for any $\theta'$ in $[0, 1)$, where

$$Z' = \left\{ c_1 v_1 + \dots + c_k v_k \colon c_1, \dots, c_k \in \mathbb{C}, \, |c_1|^2 + \dots + |c_k|^2 = 1 \right\}.$$

Now, the map from the complex $k$-sphere $S$ to $\mathbb{R}$ defined by $(c_1, \dots, c_k) \mapsto \|c_1 v_1 + \dots + c_k v_k\|_{L^2(\Omega)}$ is continuous and so achieves its minimum, $a$, which is the same as the minimum of $\|u\|_{L^2(\Omega)}$ on $Z'$. Because $(v_1, \dots, v_k)$ is independent, $a$ must be positive. Similarly, $\|u\|_Y$ achieves its maximum, $b > 0$, on $Z'$.

Thus, on $Z'$ and so on $Z$, for any $\theta' > \theta$,

$$\begin{aligned} R_V^{\theta'}(u) - R_V^\theta(u) &= \frac{(f(\theta') - f(\theta)) \int_\Gamma |u|^2}{\|u\|_{L^2(\Omega)}^2} \le C a^{-2} \|u\|_Y^2 \, (f(\theta') - f(\theta)) \\ &\le C a^{-2} b^2 (f(\theta') - f(\theta)), \end{aligned}$$

where we used the standard trace inequality for $u$ in $Y$,

$$\|u\|_{L^2(\Gamma)} \le C \|u\|_{L^2(\Omega)}^{1/2} \|\nabla u\|_{L^2(\Omega)}^{1/2},$$

followed by Poincare's inequality. But this shows that

$$\overline{N}_V^{\theta'}(\lambda) \ge \overline{N}_V^\theta(\gamma_k(\theta))$$

for $\lambda = \gamma_k(\theta) + C a^{-2} b^2 (f(\theta') - f(\theta))$. Since we already know that $\gamma_k(\theta') > \gamma_k(\theta)$ it follows that

$$\left| \gamma_k(\theta') - \gamma_k(\theta) \right| \le C a^{-2} b^2 (f(\theta') - f(\theta)),$$

meaning that $\gamma_k$ is continuous on $[0, 1)$. $\square$

The first part of Theorem 8.11 is Theorem 1.2.

**Theorem 8.11.** *Assume that $\Gamma$ is $C^2$ and has a finite number of components. For all $j$ in $\mathbb{N}$, the function $\gamma_j : [0, 1] \to [\lambda_j, \nu_j]$ is a strictly increasing continuous bijection. Also, Equation (7.2) holds for any eigenfunction of $A_V$.*

*Proof.* For any value of $\theta$ in $(0, 1)$ we let $w = w(\theta)$ be any eigenfunction of $A_V$ with eigenvalue $\gamma_j(\theta)$, normalized so that $\|w\|_H = \|w\|_{L^2(\Omega)} = 1$. We know from Proposition 8.10 that $\gamma_j(\theta)$ strictly increases continuously from $\lambda_j$ at $\theta = 0$ and remains bounded by $\nu_j$. Formally, as $\theta \to 1$, $w$ becomes an eigenfunction of $A_S$, since $w$ must approach zero on the boundary so that $\omega(w) = (\theta/(1-\theta)) w \cdot \boldsymbol{\tau}$ can remain finite. We now make this formal argument rigorous.

Letting $z = f = \theta/(1 - \theta)$, we have

$$\|w\|^2_{L^2(\Gamma)} = \int_\Gamma (w \cdot \boldsymbol{\tau})(\overline{w} \cdot \boldsymbol{\tau}) = -z^{-1} \int_\Gamma \omega(w) \overline{w} \cdot \boldsymbol{\tau},$$

the boundary integral being well-defined because of Proposition 8.8. Then

$$\int_\Gamma \omega(w) \overline{w} \cdot \boldsymbol{\tau} = -z \, \|w\|^2_{L^2(\Gamma)} \le 0,$$

so Equation (7.2) holds.

Moreover, from Definition 8.2,

$$\|\omega(w)\|^2_{L^2(\Omega)} + z \, \|w\|^2_{L^2(\Gamma)} = \gamma_j(\theta) \, \|w\|^2_{L^2(\Omega)} = \gamma_j(\theta).$$

From this we conclude two things. First, that

$$\|w\|^2_{L^2(\Gamma)} = \frac{\gamma_j(\theta) - \|\omega(w)\|^2_{L^2(\Omega)}}{z} \le \frac{\nu_j}{z}, \tag{8.7}$$

since $\gamma_j(\theta) < \nu_j$. Second, that $\|\omega(w)\|_{L^2(\Omega)} \le \gamma_j(\theta)^{1/2}$ and hence, because $\gamma_j(\theta) < \nu_j$ and by virtue of Corollary 2.15, that $\|w\|_Y \le C$.

Now letting the parameter $\theta$ vary over the set $\{1 - 1/n \colon n \in \mathbb{N}\}$, we obtain a sequence $(u^n)$ of eigenfunctions of $A_V$, $u^n = w(1 - 1/n)$, with eigenvalues $\gamma^n = \gamma_j(1 - 1/n)$. By the observations above, $(u^n)$ is a bounded sequence in $Y$. But $Y$ is compactly embedded in $H$ by Lemma 2.6 (or by Corollaries 2.12 and 2.16) so there exists a subsequence of $(u^n)$ that converges strongly in $H$. Since this subsequence is bounded in $Y$, which is a separable, reflexive Banach space, taking a further subsequence, and relabeling it $(u^n)$, we conclude that $u^n \to u$ strongly in $H$ and weak$^*$ in $Y$ to some vector field $u$ in $Y$ with $\|u\|_H = 1$ (so $u$ is nonzero).

Furthermore, $\|u^n\|_{W^{1/2}(\Gamma)} \le C \, \|u^n\|_Y \le C$, so $(u^n)$ is bounded in $W^{1/2}(\Gamma)$, which is compactly embedded in $L^2(\Gamma)$, and hence extracting a subsequence and relabeling once more, we conclude that also $u^n \to u$ strongly in $L^2(\Gamma)$. But since $z \to \infty$ as $n \to \infty$, we have $u^n \to u = 0$ in $L^2(\Gamma)$ by Equation (8.7).

Then by Definition 8.2, for any $v$ in $V$,

$$(\omega(u^n), \omega(v)) - \gamma^n (u^n, v) = 0.$$

Letting $\gamma = \lim_{n\to\infty} \gamma^n$ (the limit exists because $\gamma^n$ is a bounded increasing sequence of real numbers),

$$(\omega(u^n), \omega(v)) - \gamma(u^n, v) = (\gamma^n - \gamma)(u^n, v).$$

Since $|(u^n, v)| \le \|u^n\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \le C$, the right-hand side converges to zero. Because $u^n \to u$ strongly in $L^2(\Omega)$, $(u^n, v) \to (u, v)$. Because $u^n \to u$ weak* in $Y$, $(\omega(u_n), \omega(v)) = (\nabla u_n, \nabla v) \to (\nabla u, \nabla v) = (\omega(u), \omega(v))$, where we used Corollary A.3. We conclude that

$$(\omega(u), \omega(v)) - \gamma(u, v) = 0$$

and thus that $u$ is a weak eigenfunction of $A_S$ with eigenvalue $\gamma \le \nu_j$.

What we have shown is that $\gamma_j : [0, 1] \to [\lambda_j, \nu_k]$ for some $k \le j$ and that $\gamma_j$ is strictly increasing and continuous on all of $[0, 1]$. To show that $k = j$, we first observe that if $\gamma_k(1) = \gamma_m(1) = \nu_j$ for some $k \ne m$ then the eigenvalue $\nu_j$ has multiplicity at least 2. To see this, we repeat the compactness argument above, this time choosing the original sequence of eigenvectors $(u^{k,n})_{n=1}^\infty$ and $(u^{m,n})_{n=1}^\infty$ such that $u^{k,n}$ is orthogonal in $L^2(\Omega)$ to $u^{m,n}$, which we can always do even if they lie in the same eigenspace. We showed above that $u^{k,n} \to u$ and $u^{m,n} \to w$ in $L^2(\Omega)$ for some $u$ and $w$ that are eigenvectors of $A_S$. It is elementary to see, then, that $(u, w) = 0$, which shows that $\nu_j$ has multiplicity at least two.

Similarly, the multiplicity of the eigenvalue $\nu_j$ is at least as high as the number of distinct values of $k$ for which $\gamma_k(1) = \nu_j$. This means that the total number of eigenvalues of $A_S$ including multiplicity reached by $\gamma_j(1)$ for some $j$, $1 \le j \le k$, is at least $k$. But it can be no more than $k$ since $\gamma_j(1) = \nu_m$ for some $m \le j \le k$. Thus, the first $k$ eigenvalues of $A_L$ according to multiplicity are mapped via $\gamma_j$, $j = 1, \dots, k$, into the first $k$ eigenvalues of $A_S$, which shows that $\gamma_j : [0, 1] \to [\lambda_j, \nu_j]$ for all $j = 1, \dots, k$ and hence for all $j$ in $\mathbb{N}$, since $k$ was arbitrary. $\square$

To round out the picture of how the eigenvalues for different boundary conditions compare, we consider the eigenfunctions of the negative Laplacian with Robin boundary conditions on the vorticity. For simplicity, we restrict our attention to constant coefficients, writing the boundary conditions in terms of a parameter $\theta$ lying in $[0, 1]$, and stating only the strong form:

**Definition 8.12.** An eigenfunction $\omega_j \in W_0^1(\Omega)$ of the Dirichlet Laplacian with Robin boundary conditions satisfies

$$\begin{cases} \Delta\omega_j + \eta_j\omega_j = 0 & \text{in } \Omega, \\ (1-\theta)\nabla\omega_j \cdot \mathbf{n} + \theta\omega_j = 0 & \text{on } \Gamma. \end{cases}$$

The analog for divergence-free vector fields leads to the eigenvalue problem for a Stokes operator $A_R$ with Robin boundary conditions:

**Definition 8.13.** An eigenfunction $u_j \in X^2$ of $A_R$ satisfies $A_R u_j = \lambda_j^* u_j$ or, equivalently,

$$\begin{cases} \Delta u_j + \eta_j^* u_j = \nabla p_j, \quad \operatorname{div} u_j = 0 & \text{in } \Omega, \\ u_j \cdot \mathbf{n} = 0, \quad (1-\theta)\nabla \omega_j \cdot \mathbf{n} + \theta \omega_j = 0 & \text{on } \Gamma. \end{cases}$$

A value of $\theta = 1$ gives the operator $A_L$, and $\theta = 0$ gives Neumann boundary conditions on the vorticity.

Taking the vorticity of $u_j$ in Definition 8.13 shows that a strong eigenfunctions of $A_R$ corresponds to a strong eigenfunction of the Dirichlet Laplacian with Robin boundary conditions. Also, the equivalent of Lemma 3.4 for Robin boundary conditions on $\omega$ shows that to each strong eigenfunction of the Dirichlet Laplacian with Robin boundary conditions there corresponds a strong eigenfunctions of $A_R$. Thus, there is a bijection between the eigenfunctions and eigenvalues; that is, $\eta_j^* = \eta_j$ . Moreover, $\eta_j$ is continuous on $[0, 1)$, because the bilinear form associated to Definition 8.12 (see [6]) is continuous with $\theta$.

**Proposition 8.14.** *For all $j$ in $\mathbb{N}$, $\eta_j \colon [0, 1) \to [\mu_j, \lambda_j)$ and is strictly increasing.*

*Proof.* The proof is similar to that of Proposition 8.10, making adaptations of Filonov's proof of his theorem that parallel those in the proof of Proposition 8.10. □

**Theorem 8.15.** *For all $j$ in $\mathbb{N}$, the function $\eta_j : [0, 1] \to [\mu_j, \lambda_j]$ is continuous and strictly increasing.*

*Proof.* The proof parallels that of Theorem 8.11. □

In the addendum at the end of [6], Filonov considers Robin boundary conditions as in Definition 8.12 with, in effect, $\theta$ negative. In that case, $\eta_{j+1}(\theta) < \lambda_j$ for all $j$ in $\mathbb{N}$.

For any $\theta$,

$$\begin{aligned} &\|\nabla p_j\|_{L^2(\Omega)}^2 - \int_\Gamma (\nabla \omega_j \cdot \mathbf{n}) \omega_j \\ &\quad = \|\Delta u_j\|_{L^2(\Omega)}^2 - \eta_j(\theta) \|u_j\|_{L^2(\Omega)}^2 - \int_\Omega \Delta \omega(u_j) \omega(u_j) - \|\nabla \omega(u_j)\|_{L^2(\Omega)}^2 \\ &\quad = \eta_j(\theta) \left[ \|\omega(u_j)\|_{L^2(\Omega)}^2 - \eta_j(\theta) \|u_j\|_{L^2(\Omega)}^2 \right]. \end{aligned}$$

Thus, Equation (7.2) holds for an eigenfunction of $A_L$ ($\theta = 1$), where $\nabla p_j \equiv 0$ and $\omega_j = 0$ on $\Gamma$, and fails for an eigenfunction of the Stokes operator with Neumann boundary conditions on the vorticity ($A_R$ for $\theta = 0$), where $\nabla p_j \not\equiv 0$ and $\nabla \omega_j \cdot \mathbf{n} = 0$ on $\Gamma$. For $\theta$ in $(0, 1)$ it is not clear whether Equation (7.2) holds or not, leaving open the possibility that the inequality $\lambda_{j+1} \le \nu_j$ could be proved by showing that Equation (7.2) holds for all $\theta$ in some left neighborhood of 1 for each $\lambda_j$.

In any case, for all $j$ we have the inequalities,

$$\mu_j < \eta_j(\theta) < \lambda_j < \gamma_j(\theta') < \nu_j$$

for all $\theta$, $\theta'$ in $(0, 1)$, and

$$\mu_{j+1} < \lambda_j < \beta_j < \nu_j.$$

## Appendix A. Various lemmas

Corollary A.1 is a corollary of Lemma 2.4: it is the main tool we use to prove the equivalence of the weak and strong formulations of our eigenvalue problems. The conditions in this corollary for equality to hold are the weakest possible to insure that each factor lies in the correct space for each term to be finite.

**Corollary A.1.** *Assume that $\Omega$ is a bounded domain with locally Lipschitz boundary. For any divergence-free distribution $u$ for which $\omega(u)$ is in $W^1(\Omega)$ and any $v$ in $L^2(\Omega)$ with $\omega(v)$ in $L^2(\Omega)$,*

$$(\omega(u),\, \omega(v)) = -(\Delta u, v) + \int_\Gamma \omega(u)\overline{v} \cdot \boldsymbol{\tau}.$$

*Proof.* The vector field $v$ is in $E(\Omega)$ because $v^\perp$ is in $L^2(\Omega)$ and $\operatorname{div} v^\perp = -\omega(v)$ is in $L^2(\Omega)$. Thus, $\omega(u)$ lying in $W^1(\Omega)$, we can apply Lemma 2.4 to obtain

$$(\omega(u), \omega(v)) = -(\omega(u), \operatorname{div} v^\perp) = (\nabla\omega(u), v^\perp) - \int_\Gamma \omega(u)(\overline{v}^\perp \cdot \mathbf{n}).$$

But $(\nabla\omega(u), v^\perp) = -(\nabla^\perp\omega(u), v) = (-\Delta u, v)$ and $(\overline{v}^\perp \cdot \mathbf{n}) = -\overline{v} \cdot \boldsymbol{\tau}$, from which the result follows. $\square$

**Lemma A.2.** *Assume that $\Omega$ is a bounded domain with locally Lipschitz boundary. If $u$ is in $W^1(\Omega)$ with $\operatorname{div} u = 0$ and $v$ is in $(C^1(\Omega))^2$ then*

$$(\omega(u),\, \omega(v)) = (\nabla u, \nabla v) - \int_\Gamma (\nabla u \overline{v}) \cdot \mathbf{n}.$$

*Proof.* We have,

$$\begin{aligned}
\omega(u)\omega(\overline{v}) &= (\partial_1 u^2 - \partial_2 u^1)(\partial_1 \overline{v}^2 - \partial_2 \overline{v}^1) \\
&= \partial_1 u^2 \partial_1 \overline{v}^2 + \partial_2 u^1 \partial_2 \overline{v}^1 - (\partial_1 u^2 \partial_2 \overline{v}^1 + \partial_2 u^1 \partial_1 \overline{v}^2) \\
&= \partial_1 u^2 \partial_1 \overline{v}^2 + \partial_2 u^1 \partial_2 \overline{v}^1 + \partial_1 u^1 \partial_1 \overline{v}^1 + \partial_2 u^2 \partial_2 \overline{v}^2 \\
&\qquad - (\partial_1 u^2 \partial_2 \overline{v}^1 + \partial_2 u^1 \partial_1 \overline{v}^2 + \partial_1 u^1 \partial_1 \overline{v}^1 + \partial_2 u^2 \partial_2 \overline{v}^2) \\
&= \partial_i u^j \partial_i \overline{v}^j - \partial_i u^j \partial_j \overline{v}^i = \nabla u \cdot \nabla \overline{v} - (\nabla u)^T \cdot \nabla \overline{v}.
\end{aligned}$$

Since $\operatorname{div} u = 0$, we have $(\nabla u)^T \cdot \nabla \overline{v} = \partial_i u^j \partial_j \overline{v}^i = \partial_j(\partial_i u^j \overline{v}^i) = \operatorname{div}(\nabla u \overline{v})$. But $\nabla u \overline{v}$ is in $L^2(\Omega)$ and

$$\|\operatorname{div}(\nabla u \overline{v})\|_{L^2(\Omega)} \le \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^\infty(\Omega)}$$

is finite, so $\nabla u \overline{v}$ is in $E(\Omega)$ and we can apply Lemma 2.4 to give

$$(\omega(u), \omega(v)) = (\nabla u, \nabla v) - \int_\Omega \operatorname{div}(\nabla u \overline{v}) = (\nabla u, \nabla v) - \int_\Gamma (\nabla u \overline{v}) \cdot \mathbf{n}.$$

□

**Corollary A.3.** *Assume that $\Omega$ is a bounded domain with locally Lipschitz boundary. For all $u$ in $W^1(\Omega)$ with $\operatorname{div} u = 0$ and all $v$ in $V$,*

$$(\omega(u), \omega(v)) = (\nabla u, \nabla v).$$

*Proof.* This follows from Lemma A.2 and the density of $C^1(\Omega)$ in $W^1(\Omega)$. □

**Lemma A.4.** *Assume that $\Gamma$ is $C^2$. For all $u$ in $H \cap W^2(\Omega)$ and $v$ in $Y$,*

$$\nabla u v \cdot \mathbf{n} = -\kappa u \cdot v.$$

*Proof.* Because $u \cdot \mathbf{n}$ has a constant value (of zero) along $\Gamma$,

$$0 = \frac{\partial}{\partial \boldsymbol{\tau}}(u \cdot \mathbf{n}) = \frac{\partial u}{\partial \boldsymbol{\tau}} \cdot \mathbf{n} + u \cdot \frac{\partial \mathbf{n}}{\partial \boldsymbol{\tau}} = \nabla u \boldsymbol{\tau} \cdot \mathbf{n} + \kappa u \cdot \boldsymbol{\tau}.$$

But $v = (v \cdot \boldsymbol{\tau})\boldsymbol{\tau}$, so multiplying both sides of the above inequality by $v \cdot \boldsymbol{\tau}$ completes the proof. □

**Lemma A.5.** *Assume that $\Gamma$ is $C^2$. For all $u$ in $H \cap W^2(\Omega)$ and $v$ in $Y$*

$$\nabla u \mathbf{n} \cdot v = \omega(u) v \cdot \boldsymbol{\tau} - \kappa u \cdot v.$$

*Proof.* Writing

$$\mathbf{n} = \begin{pmatrix} n^1 \\ n^2 \end{pmatrix}, \quad \boldsymbol{\tau} = \begin{pmatrix} -n^2 \\ n^1 \end{pmatrix}$$

with $(n^1)^2 + (n^2)^2 = 1$, we have

$$\begin{aligned}
&\nabla u \mathbf{n} \cdot \boldsymbol{\tau} - \nabla u \boldsymbol{\tau} \cdot \mathbf{n} \\
&\quad = \left( \begin{pmatrix} \partial_1 u^1 & \partial_2 u^1 \\ \partial_1 u^2 & \partial_2 u^2 \end{pmatrix} \begin{pmatrix} n^1 \\ n^2 \end{pmatrix} \right) \cdot \begin{pmatrix} -n^2 \\ n^1 \end{pmatrix} - \left( \begin{pmatrix} \partial_1 u^1 & \partial_2 u^1 \\ \partial_1 u^2 & \partial_2 u^2 \end{pmatrix} \begin{pmatrix} -n^2 \\ n^1 \end{pmatrix} \right) \cdot \begin{pmatrix} n^1 \\ n^2 \end{pmatrix} \\
&\quad = \begin{pmatrix} \partial_1 u^1 n^1 + \partial_2 u^1 n^2 \\ \partial_1 u^2 n^1 + \partial_2 u^2 n^2 \end{pmatrix} \cdot \begin{pmatrix} -n^2 \\ n^1 \end{pmatrix} - \begin{pmatrix} -\partial_1 u^1 n^2 + \partial_2 u^1 n^1 \\ -\partial_1 u^2 n^2 + \partial_2 u^2 n^1 \end{pmatrix} \cdot \begin{pmatrix} n^1 \\ n^2 \end{pmatrix} \\
&\quad = -\partial_1 u^1 n^1 n^2 - \partial_2 u^1 (n^2)^2 + \partial_1 u^2 (n^1)^2 + \partial_2 u^2 n^1 n^2 \\
&\qquad\quad + \partial_1 u^1 n^1 n^2 - \partial_2 u^1 (n^1)^2 + \partial_1 u^2 (n^2)^2 - \partial_2 u^2 n^1 n^2 \\
&\quad = \left[(n^1)^2 + (n^2)^2\right] \left[\partial_1 u^2 - \partial_2 u^1\right] = \omega(u).
\end{aligned}$$

Thus by Lemma A.4,

$$\nabla u \mathbf{n} \cdot \boldsymbol{\tau} = \omega(u) + \nabla u \boldsymbol{\tau} \cdot \mathbf{n} = \omega(u) - \kappa u \cdot \boldsymbol{\tau},$$

and multiplying both sides by $v \cdot \boldsymbol{\tau}$ completes the proof. □

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## References

[1] Robert A. Adams. *Sobolev spaces*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65. 4, 5, 6, 7, 10

[2] Mark S. Ashbaugh. On universal inequalities for the low eigenvalues of the buckling problem. In *Partial differential equations and inverse problems*, volume 362 of *Contemp. Math.*, pages 13–31. Amer. Math. Soc., Providence, RI, 2004. 2, 12

[3] Russell M. Brown and Zhongwei Shen. Estimates for the Stokes operator in Lipschitz domains. *Indiana Univ. Math. J.*, 44(4):1183–1206, 1995. 12

[4] Thierry Clopeau, Andro Mikelić, and Raoul Robert. On the vanishing viscosity limit for the 2D incompressible Navier-Stokes equations with the friction type boundary conditions. *Nonlinearity*, 11(6):1625–1636, 1998. 3, 25

[5] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998. 6

[6] N. Filonov. On an inequality for the eigenvalues of the Dirichlet and Neumann problems for the Laplace operator. *Algebra i Analiz*, 16(2):172–176, 2004. 2, 18, 30

[7] C. Foias, O. Manley, R. Rosa, and R. Temam. *Navier-Stokes equations and turbulence*, volume 83 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2001. 11

[8] Leonid Friedlander. Some inequalities between Dirichlet and Neumann eigenvalues. *Arch. Rational Mech. Anal.*, 116(2):153–160, 1991. 2, 3

[9] Leonid Friedlander. Remarks on the membrane and buckling eigenvalues for planar domains. *Mosc. Math. J.*, 4(2):369–375, 535, 2004. 1

[10] Emilio Gagliardo. Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in $n$ variabili. *Rend. Sem. Mat. Univ. Padova*, 27:284–305, 1957. 4

[11] Giovanni P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I*, volume 38 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, New York, 1994. 4, 5, 6, 7, 9, 10

[12] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Springer-Verlag, Berlin, 1977. Grundlehren der Mathematischen Wissenschaften, Vol. 224. 4

[13] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985. 4

[14] Antoine Henrot. *Extremum problems for eigenvalues of elliptic operators*. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006. 14

[15] Dragoş Iftimie and Gabriela Planas. Inviscid limits for the Navier-Stokes equations with Navier friction boundary conditions. *Nonlinearity*, 19(4):899–918, 2006. 3

[16] Dragoş Iftimie and Franck Sueur. Viscous boundary layers for the Navier-Stokes equations with the Navier slip conditions. *Preprint*. 3

[17] James P. Kelliher. Navier-Stokes equations with Navier boundary conditions for a bounded domain in the plane. *SIAM Math Analysis*, 38(1):210–232, 2006. 3, 22

[18] S. Kesavan. *Topics in functional analysis and applications*. John Wiley & Sons Inc., New York, 1989. 6

[19] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod, 1969. 12, 13

[20] Pierre-Louis Lions. *Mathematical topics in fluid mechanics. Vol. 1*, volume 3 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press Oxford University Press, New York, 1996. 12, 13

[21] M. C. Lopes Filho, H. J. Nussenzveig Lopes, and G. Planas. On the inviscid limit for 2d incompressible flow with Navier friction condition. *SIAM Math Analysis*, 36(4):1130 – 1141, 2005. 3

[22] V. N. Maslennikova and M. E. Bogovskiĭ. Approximation of potential and solenoidal vector fields. *Sibirsk. Mat. Zh.*, 24(5):149–171, 1983. 7

[23] L. E. Payne. Inequalities for eigenvalues of membranes and plates. *J. Rational Mech. Anal.*, 4:517–529, 1955. 1

[24] L. E. Payne. Isoperimetric inequalities and their applications. *SIAM Rev.*, 9:453–488, 1967. 1, 12

[25] Roger Temam. *Navier-Stokes equations*. AMS Chelsea Publishing, Providence, RI, 2001. Theory and numerical analysis, Reprint of the 1984 edition. 6, 7, 12, 16

[26] Alexander Weinstein and William Stenger. *Methods of intermediate problems for eigenvalues*. Academic Press, New York, 1972. Theory and ramifications, Mathematics in Science and Engineering, Vol. 89. 26

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