1. Introduction

1.1. Active transport equations. We consider active transport equations of the form,

\[
\begin{cases}
\partial_t \rho + v \cdot \nabla \rho = F(\rho), \\
v = K * \rho,
\end{cases}
\]

in \( \mathbb{R}^d, d \geq 2 \). We assume that the function \( F: \mathbb{R} \to \mathbb{R} \) is continuously differentiable with \( F' \) locally Lipschitz and \( F(0) = 0 \). To fix terminology, we refer to \( \rho \) as the density of an unspecified substance (though we impose no requirement on the sign of \( \rho \)) and \( v \) as the velocity field.

Throughout this paper we fix \( \alpha \in (0,1) \).
Our purpose is to show that if (subsets of) the level sets of the initial density (that is, sets on which \( \rho_0 \) is constant) have \( C^{1, \alpha} \) regularity as a curve in 2D, surface in 3D, or hypersurface for \( d \geq 4 \), then they retain that regularity for the existence time of the weak solution. Hence, we employ a kernel that neither smooths the density nor introduces additional singularities, as smoothing would trivialize the problem and adding singularities would prevent the propagation of level set regularity. In short, this means that the kernel must impart to \( v \) one more derivative of regularity than that possessed by \( \rho \). A not quite completely general form of such a kernel is

\[
K = R \nabla \Phi,
\]

where \( \Phi \) is the fundamental solution of the Laplacian, or Newtonian potential (so \( \Delta \Phi = \delta \)), and \( R \) is a rotation matrix.

In the 2D special case in which \( F \equiv 0 \) and \( R \) gives rotation counterclockwise by 90° (so \( K = \nabla^\perp \Phi \)), we obtain the 2D Euler equations without forcing:

\[
\begin{align*}
\partial_t \omega + v \cdot \nabla \omega &= 0, \\
v &= \nabla^\perp \Phi * \omega, \\
\omega(0) &= \omega_0,
\end{align*}
\]

where we write \( \omega \) in place of \( \rho \), and where \( \nabla^\perp := (-\partial_2, \partial_1) \). Here, \( \omega = \text{curl } v := \partial_1 v^2 - \partial_2 v^1 \) is the vorticity and \( \text{div } v = 0 \). This was the first instance of an active scalar equation for which the propagation of striated regularity, a generalization of the level set regularity in this paper, was demonstrated—by Chemin [7, 8]. This gave as a special case that the regularity of the boundary of a vortex patch (initial vorticity the characteristic function of a bounded domain) is propagated over time (also shown in [2]).

1.2. Aggregation equation. References [7, 8, 20] and all subsequent studies of the vortex patch problem ([11, 14, 12, 1], for instance) very much rely upon \( v \) being divergence-free, both to derive identities related to transport equations and to obtain estimates in negative Hölder spaces of the regularity of solutions to the resulting transport equations, estimates that do not hold when \( \text{div } v \neq 0 \). In this paper, we treat the complementary case corresponding to \( R = -I \), in which \( \text{curl } v = 0 \) but \( \text{div } v \neq 0 \), choosing

\[
K = -\nabla \Phi.
\]

Hence, we have a type of gradient flow, in which \( \text{div } v = -\rho \).

This is the kernel appearing in the aggregation equation with Newtonian potential (the inviscid form of a limiting case of the Keller-Segel equation),

\[
\begin{align*}
\partial_t \rho + v \cdot \nabla \rho &= \rho^2, \\
v &= -\nabla \Phi * \rho, \\
\rho(0) &= \rho_0.
\end{align*}
\]

Then \( \text{div}(\rho v) = v \cdot \nabla \rho + \rho \text{div } v = v \cdot \nabla \rho - \rho^2 \), so (1.5) can be written in the usual divergence form, \( \partial_t \rho + \text{div}(\rho v) = 0 \). We see that \( F(\rho) = \rho^2 \) in (1.5), but we work with a general \( F \), as it creates no significant additional difficulties and, in fact, the structure of the associated transport equations are a little clearer in the more general form. It also makes more transparent the comparison to the approach in [3], in which (1.5) is transformed into (1.1, 1.4) with \( F = \rho(\rho - 1) \equiv 0 \) for patch data (see Section 11).
To get a feel for the problem, let us consider the situation somewhat formally at first. Let us assume that \( v \) possesses a unique flow map, \( \eta \); that is, \( \eta: [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \) satisfies
\[
\partial_t \eta(t, x) = v(t, \eta(t, x)), \quad \eta(0, x) = x.
\]
Then \((1.1)_{1,3}\) can be written as
\[
\frac{d}{dt} \rho(t, \eta(t, x)) = F(\rho(t, \eta(t, x))), \quad \rho(0, x) = \rho_0(x)
\]
or in integral form as
\[
\rho(t, \eta(t, x)) = \rho_0(x) + \int_0^t F(\rho(s, \eta(s, x))) \, ds.
\]

Fixing \( x \in \mathbb{R}^d \) and defining \( \varphi(t) = \rho(t, \eta(t, x)) \), we have \( \varphi'(t) = F(\varphi(t)), \varphi(0) = \rho_0(x) \). Since \( F \) is locally Lipschitz (we will also need \( F' \) locally Lipschitz later in the proof of Proposition 8.2), there exists a unique solution at least up to the possibly infinite time,
\[
T^\ast = T^\ast(||\rho_0||_{L^\infty}, F).
\]
If \( F \geq 0 \) then it is easy to see that \( T^\ast \) is a decreasing function of its first argument.

For example, when \( F(\rho) = \rho^2 \) as in \((1.5)\), we find (see, for instance, the discussion in \([4]\)),
\[
\varphi(t) = \rho(t, \eta(t, x)) = \frac{\rho_0(x)}{1 - \rho_0(x)t},
\]
and \( T^\ast = ||\rho_0^+||_{L^\infty}^{-1} \), where \( \rho_0^+ \) is the positive part of \( \rho_0 \). Solutions are global-in-time if \( \rho_0 \leq 0 \).

These formal considerations can be made rigorous and lead to Theorem 1.2, below.

**Definition 1.1.** Fix \( T > 0 \) and let \( v: [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \) be a velocity field that is log-Lipschitz in space. Let \( \eta: [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \) be its unique flow map (as in \((1.6))\). We say that \( \rho \) is a Lagrangian solution to \((1.1, 1.4)\) with initial density \( \rho_0 \) if \((1.8)\) holds and \( \rho(t, x) = -\text{div}v(t, x) \) for all \((t, x) \in [0, T] \times \mathbb{R}^d\).

Existence of weak solutions for the particular kind of striated regularity we will impose on the initial data will appear as part of the proof of Theorem 2.3. In its proof, however, we use the following well-posedness result, applied to smooth initial data.

**Theorem 1.2.** Let \( \rho_0 \in L^\infty(\mathbb{R}^d) \) have compact support, \( K_0 \subseteq \mathbb{R}^d \). Fix \( T > 0 \) with \( T < T^\ast \). There exists a unique Lagrangian solution to \((1.1, 1.4)\), and \( \rho(t) \) remains compactly supported in some \( K(T) \subseteq \mathbb{R}^d \), with \( K(T) \) depending only upon \( T, K_0 \), and \( ||\rho_0||_{L^\infty} \). Moreover,
\[
||\rho(t)||_{L^p} \leq C ||\rho_0||_{L^p}
\]
for all \( p \in [1, \infty] \) and \( t \in (0, T) \), where \( C = C(T, p, K_0, ||\rho_0||_{L^\infty}) \).

If \( \rho_0 \) also lies in \( C^{k,\alpha}(\mathbb{R}^d) \) for some \( k \geq 0 \) then \( \rho \in L^\infty(0, T; C^{k,\alpha}(\mathbb{R}^d)) \). When \( k \geq 1, \rho \in C^k(0, T; C^{k,\alpha}(\mathbb{R}^d)) \) and is also the unique, classical Eulerian solution.

**Proof.** The proof in the case of the aggregation equation \((F(\rho) = \rho^2)\) is given in \([4]\); for a slightly more general case, see \([10]\). Both proofs are easily adapted to allow the general form in \((1.1)_1\). We note here only that we imposed the restriction that \( F(0) = 0 \) so that the compact support of \( \rho(t) \) will be maintained for the full time of existence. \( \square \)

We also have the following formulation of a weak Eulerian solution to \((1.1, 1.4)\):
Definition 1.3. We say that \( \rho \in C([0,T]; L^2(\mathbb{R}^d)) \) is a weak Eulerian solution to (1.1, 1.4) with initial density \( \rho_0 \in L^2(\mathbb{R}^d) \) if \( \rho(0) = \rho_0 \) and
\[
\int_{(0,T) \times \mathbb{R}^d} (\rho \partial_t \varphi + \rho v \cdot \nabla \varphi + (F(\rho) - \rho^2) \varphi) = 0
\]
for all \( \varphi \in C_c^\infty((0,T) \times \mathbb{R}^d) \). That is, \( \rho \) satisfies (1.1, 1.4) as a distribution on \((0,T) \times \mathbb{R}^d\).

2. Main results

By patch data in the context of active scalar equations we mean that the active scalar, \( \rho \) in the case of (1.1), is the characteristic function of a domain. The interest in “patch problems,” by which we mean the question of whether the regularity of the patch boundary is maintained over time, seems to have originated in [22], where Yudovich first formulated it for the 2D Euler equations. Interest was renewed by the survey paper [16], and the question was answered affirmatively for the 2D Euler equations by Chemin in [7, 8].

Actually, Chemin did much more in [7, 8] than address the patch problem. He showed, roughly speaking, that initial vorticity having \( C^\alpha \) regularity in a set of directions measured by a family of \( C^\alpha \) vector fields that together foliate the plane maintain \( C^\alpha \) regularity in the directions of the pushed forward vector fields, which themselves maintain their \( C^\alpha \) regularity. This kind of regularity he called striated. In the case of patch data for a \( C^{1,\alpha} \) boundary, the key \( C^\alpha \) vector field would be a tangent vector field on the boundary, extended somewhat arbitrarily to be \( C^\alpha \) throughout the plane. The pushforward of this vector field would remain tangent to the boundary of the patch, and hence imply the \( C^{1,\alpha} \) regularity of the patch boundary for all time.

Ideally, we would prove a result for the aggregation equation as general as that proved by Chemin for the 2D Euler equations. This is not, however, possible for reasons that we will explain subsequently. What we can show, however, is the following more limited result (stated more precisely and more fully in Theorem 2.3):

**Main result roughly stated:** Assume that the level sets of \( \rho_0 \) are composed of level sets of \( C^{1,\alpha} \) functions, \( \psi_1, \ldots, \psi_N \), which are parallel to the boundary, to give the \( C^{1,\alpha} \) regularity of the patch. If there are multiple patches, possibly nested, then we simply use one such function for each patch.

So in Theorem 2.3, we assume there are \( N \) non-overlapping domains of irregularities in \( \rho_0 \) characterized in this manner by \( \psi_1, \ldots, \psi_N \), but on the remainder of the plane, \( \rho_0 \) is \( C^{1,\alpha} \). A difficulty that arises, however, is that the transported functions, \( \psi_1(t), \ldots, \psi_N(t) \), are only Lipschitz continuous, even though their level sets turn out to be \( C^{1,\alpha} \). To obtain this regularity, we need to introduce another set of vector fields, \( Z_1, \ldots, Z_N \), which are parallel to \( \nabla \psi_j \) for all time and which remain \( C^\alpha \). As applied to the patch problem, then, we use the \( C^\alpha \) regularity of each \( Z_j \), which will be normal to the patch boundary, to give the \( C^{1,\alpha} \) regularity of the boundary.

A precise statement of all this is in Definition 2.1 and Theorem 2.3.

Definition 2.1. Let \( f, g \in L^\infty(\mathbb{R}^d) \) and \( U \) be an open subset of \( \mathbb{R}^d \). Recalling that measurable functions are really equivalence classes of functions defined only up to sets of measure 0, we
say that \( f \) is level-set-compatible with \( g \) on \( U \) if for some representatives of their respective equivalence classes, which we will continue to call \( f \) and \( g \), each level set of \( f \) in \( U \) is contained in a level set of \( g \) in \( U \), up to sets of measure zero. (So, for instance, any \( f \) is compatible with a constant function.)

**Definition 2.2.** We say that an initial compactly supported density \( \rho_0 \in L^\infty(\mathbb{R}^d) \) is suitable if there exists a collection \( (\psi_j)_{j=1}^N \) of \( C^{1,\alpha}(\mathbb{R}^d) \)-functions with the following properties:

- Each \( \psi_j \) is level-set-compatible with \( \rho_0 \) on \( U_j \), the interior of \( \text{supp} \, \psi_j \).
- For each \( j \leq N \), \( U_j \) is connected and intersects no other \( U_k \).
- There exists an open \( V \subseteq \mathbb{R}^d \), which, together with the \( U_j \)'s, cover \( \mathbb{R}^d \).
- On \( V \), \( \rho_0 \) is \( C^{1,\alpha} \).
- There exists \( r_0 > 0 \) such that for all \( j \) and all \( x \in \partial U_j \), the ball \( B_{r_0}(x) \subseteq V \).
- \( |\nabla \psi_j| \geq C_0 \) on \( U_j \setminus V \) for some fixed \( C_0 > 0 \).

**Theorem 2.3.** Let \( \rho_0 \) be a suitable initial density, as in Definition 2.2, and let \( \rho \) be the unique Lagrangian solution to (1.1, 1.4) given by Theorem 1.2 on \([0,T]\) for some \( T < T^* \). Then \( \rho \) is also a weak Eulerian solution to (1.1, 1.4) as in Definition 1.3 and the following hold:

(a) \( \nabla v \in L^\infty((0,T) \times \mathbb{R}^d) \) with \( \|\nabla v(t)\|_{L^\infty} \leq Ce^{Ct} \), where \( C \) depends only upon \( \rho_0 \).

(b) Let \( \psi_j(t) \) be \( \psi_j \) transported by the flow map for \( v \). Then for all \( t \in [0,T] \),

\[
\|\nabla \psi_j(t)\|_{L^\infty} \leq \|\nabla \psi_j(0)\|_{L^\infty} e^{C(t)}t ,
\]

\( \psi_j(t) \) is level-set-compatible with \( \rho(t) \) on the interior of \( \text{supp} \, \psi_j(t) \) and the level sets of \( \psi_j(t) \) are \( C^{1,\alpha} \). Here, \( C(T) \) depends continuously on \( F \), \( \rho_0 \), and \( T \).

(c) For each \( j \) there exists a solution \( Z_j \in L^\infty(0,T;C^\alpha(\mathbb{R}^d)) \) to

\[
\partial_t Z_j + v \cdot \nabla Z_j = -Z_j \cdot \nabla v - \rho Z_j, \quad Z_j(0) = Y_{0,d}^{(j)} ,
\]

and \( Z_j(t) \) is non-vanishing and parallel to \( \nabla \psi_j(t) \) for all \( t \in [0,T] \). Here, \( Y_{0,d}^{(j)} = \wedge_{i<d} Y_{0,i}^{(j)} \) is defined in Section 3 using a sufficient family of vector field in Definition 4.2.

If \( \rho_0 = c \mathbb{1}_\Omega \), then we see that \( \rho(t) = c(t) \mathbb{1}_\Omega_t \), where \( \Omega_t = \eta(t, \Omega) \) and the value of \( c(t) \) is derived from (1.7); that is, a patch remains a patch over time. Corollary 2.4 shows that the boundary of a patch maintains \( C^{1,\alpha} \) regularity, extending the result in [3] to a general \( F \).

**Corollary 2.4.** Assume that \( \rho_0 = c \mathbb{1}_\Omega \) for some constant \( c \), where \( \Omega \) is a bounded domain in \( \mathbb{R}^d \) with \( C^{1,\alpha} \) boundary and having a finite number of connected components. Let \( \rho = c(t) \mathbb{1}_\Omega_t \) be the unique Lagrangian solution to (1.1, 1.4) given by Theorem 1.2. Then \( \partial \Omega_t \) is \( C^{1,\alpha} \) for all \( t \in (0,T^*) \).

Were \( \rho_0 \in C^{1,\alpha} \) on all of \( \mathbb{R}^d \), Theorem 1.2 would give \( \nabla v(t) \in C^\alpha(\mathbb{R}^d) \) for all time, whereas for the solutions given by Theorem 2.3, \( \nabla v(t) \) is only in \( L^\infty(\mathbb{R}^d) \). Theorem 2.5 shows that \( \nabla v(t) \) is \( C^{1,\alpha} \) in the same directions in which \( \rho(t) \) is \( C^{1,\alpha} \). For patch data, this is no surprise, since \( \nabla v \) is \( C^\infty \) inside and outside the patch, but for more general initial data it is perhaps somewhat surprising, since \( v \) is recovered from \( \rho \) globally by (1.1)2, and physically one views the active scalar \( \rho \) as carrying the irregularities in the solution.

A version of Theorem 2.5 for 2D Euler is due to Serfati [20].

**Theorem 2.5.** Let \( \rho, v, \eta \) be as in Theorem 2.3. There exists a matrix field \( A \in L^\infty(0,T;C^\alpha(\mathbb{R}^d)) \) such that for all \( t \in [0,T] \),

\[
\nabla v(t) - \rho(t) A(t) \in C^\alpha(\mathbb{R}^d) .
\]
The matrix $A$ of Theorem 2.5 has a very simple form: see Remark 10.1.

From Theorems 2.3 and 2.5 we will establish Theorem 2.6.

**Theorem 2.6.** Let $\rho$ be the solution to $(1.1, 1.4)$ given in Theorem 2.3. If $\rho_0 \in C^{k,\alpha}(W)$ for some open $W \subseteq \mathbb{R}^d$, where $k = 0$ or $1$, then $\rho(t) \in C^{k,\alpha}(\eta(t,W))$ for all $t \in [0,T]$ with a norm bounded over $[0,T]$.

2.1. **Overview.** A logical starting point for any analysis of the patch problem or its generalizations is to describe the level sets using a function that has greater regularity than $\rho_0$ itself (which may have no regularity in directions perpendicular to its level sets), transport this function by the flow map, and use its regularity to indicate the regularity of the level sets at time $t$. In Theorem 2.3, this could be $\psi_j$, which has $C^{1,\alpha}$ regularity, except that $\nabla \psi_j(t)$ fails to have $C^\alpha$ regularity for $t > 0$.

For an aggregation patch, this difficulty is overcome in [3] by adding a carefully chosen forcing term to the transport equation for $\psi_j$, which leads to an evolution equation for $\nabla \psi_j$. The special forcing term cancels a singularity that would otherwise appear in this evolution equation and allows the authors of [3] to show that the $C^{1,\alpha}$ regularity of the boundary persists over time. (In Section 11 we give a more in-depth explanation.)

As we will see, the evolution equation in [3] is actually that of the wedge product of $d - 1$ linearly independent vector fields initially tangent to the patch boundary that are each separately pushed forward by the flow map. This is in accordance with the approach taken by Chemin for the 2D Euler equations in [8] and in higher dimension by Danchin in [11], used to eliminate the apparent singularity that occurs in the evolution equation for $\nabla \psi_j$.

What allows the approach in [8, 11] to work is that the pushforward of any vector field that is tangent to the level sets initially remains tangent to the transported level sets for all time. Hence, if we construct a complete set of vector fields, $(Y_1, \ldots, Y_{d-1})$, tangent to the level sets, we can push these forward by the flow to serve as a framework against which to assess the regularity of the level sets. Topological considerations make it clear that one set of vector fields will usually not suffice, but rather an entire family will be needed. The idea of using such a family goes back to Bony [5] (see the remarks at the end of Chapter 5 of [9]).

In treating the initial data, however, we will encounter a substantial complication that does not appear for the Euler equations, and it is this complication that makes it impossible to treat irregularities in the initial data as general as those treated for the Euler equations.

To explain this complication, we must first define what we mean by $Y \cdot \nabla \rho$ when $\rho$ lies only in $L^\infty$ with $Y \in C^\alpha(\mathbb{R}^d)$ and $\text{div } Y \in L^\infty(\mathbb{R}^d)$, so $Y \cdot \nabla \rho$ does not make sense as a pointwise product. We interpret this product in the sense of distributions as

$$Y \cdot \nabla \rho := \text{div}(\rho Y) - \rho \text{div } Y.$$  

On the right-hand side, $\text{div}(\rho Y)$ is a distributional derivative (and so a distribution) and $\rho \text{div } Y$ is a regular distribution. Equivalently, we can define $Y \cdot \nabla \rho$ by its action on test functions:

$$(Y \cdot \nabla \rho, \varphi) = -(\rho, \text{div}(\varphi Y))$$ for all $\varphi \in C^\infty_c(\mathbb{R}^d).$ \hspace{1cm} (2.4)

Now, suppose that $Y \cdot \nabla \rho = 0$. We would like to interpret this as meaning that $\rho$ is constant a.e. in directions given by $Y$, as it would if $Y$ and $\rho$ were sufficiently smooth: the unique flow map for $Y$ would determine its directions and $Y \cdot \nabla \rho = 0$ would mean that $\rho$ is constant along the flow lines. (Or, the flow lines form level sets of $\rho$.) But $Y$ is only $C^\alpha$, so flow maps exist for it but need not be unique. And even were $Y$ smooth enough to give
unique flow lines, \( Y \cdot \nabla \rho = 0 \) would only mean that \( (\rho, \text{div}(\varphi Y)) = 0 \) for all \( \varphi \in C^\infty_c(\mathbb{R}^d) \), and it is not easy to show that this means that \( \rho \) is constant a.e. along the flow lines.

The true difficulty with \( Y \cdot \nabla \rho \), though, lies not in its interpretation, but in preparing the initial data in such a way that \( Y_0 \cdot \nabla \rho_{0,n} \)—where \((\rho_{0,n})\) is a sequence of approximating, more regular initial densities—is controlled by \( Y_0 \cdot \nabla \rho_0 \) in the appropriate norm. In treating the 2D Euler equations (1.3) in [7, 8], the appropriate norm is the negative Hölder space, \( C^{a,-1} \), which allows \( \omega_{0,n} \) to be prepared in the standard way by mollification, leading easily to the bound, \( \|\text{div}(\omega_{0,n} Y_0)\|_{C^{a,-1}} \leq C \|\text{div}(\omega_0 Y_0)\|_{C^{a,-1}} \). This norm is appropriate, because \( \text{div}(\omega_0 Y_0) \) is purely transported by the flow (so it is used instead of \( Y_0 \cdot \nabla \omega_0 \)), and the velocity field is divergence-free, which allows the propagation of regularity in the \( C^{a,-1} \) norm.

As we will see in Section 8, however, neither \( Y_0 \cdot \nabla \rho_0 \) nor \( \text{div}(\rho_0 Y_0) \) are purely transported by the flow for (1.1, 1.4). We can easily propagate the \( L^\infty \) norm of \( Y_0 \cdot \nabla \rho_0 \) and \( \text{div} Y_0 \), but simple mollification of \( \rho_0 \) to obtain \( \rho_{n,0} \) will not lead to any control on \( Y_0 \cdot \nabla \rho_{n,0} \) in \( L^\infty \), so this is of no use. Conceptually, we avoid these difficulties by using a family of \( C^{1,a} \) hypersurfaces, both to regularize the initial data so that \( \rho_{0,n} \in C^{1,a} \) and \( Y_0 \cdot \nabla \rho_{0,n} \) is bounded in \( L^\infty \), and to construct an appropriate family of vector fields against which to measure the striated regularity as they are pushedforward by the flow map. In actuality, we never define these hypersurfaces, using only their normal vector fields given by the family of functions \( (\nabla \psi_j) \).

Beyond preparing the initial data, the core difficulty in proving the propagation of striated regularity, even for patch data, is showing that the velocity gradient remains Lipschitz over time. Although our manner of proof is adapted from [7, 8], this core difficulty is handled, as it is in [1], by adapting Serfati’s [20]. This appears in our use of Lemma 5.1 to establish (9.5), the key inequality that closes the bounds that yield \( \nabla v \in L^\infty((0,T) \times \mathbb{R}^d) \).

Finally, we mention that in 2D, if we assume that \( \text{div} Y_0 = 0 \), a unique classical flow map for \( Y_0 \) does exist and the corresponding transport equation has a unique solution, as shown in [6]. (Such uniqueness does not extend to higher dimension or to non-autonomous systems.) This would allow us to prepare the initial data in 2D under the assumption that \( Y_0 \in C^a(\mathbb{R}^2) \), \( \text{div} Y_0 = 0 \), \( \rho_0 \in L^\infty(\mathbb{R}^2) \), and \( Y_0 \cdot \nabla \rho_0 = 0 \). Here, \( \mathcal{X} \) is a collection of vector fields, a sufficient family, as defined as in [8, 11] (or see Definition 4.1, below). We do not pursue this in detail, here, however, as we are interested in a dimension-independent result.

### 2.2. Organization of this paper.

In Section 3, we fix some notation and make a few definitions, leaving the definition of a sufficient family to Section 4. In Section 5, we give a linear algebra lemma due to Serfati that will be used in our proof of Theorem 2.3 to obtain a refined estimate on \( \nabla v \) in \( L^\infty \). Section 6 includes a number of lemmas centered around \( \nabla v \). In Section 7, we discuss the preparation of the initial data. In Section 8, we derive the transport equations for the pushforward of a vector field by a gradient vector field. In Section 9, we prove Theorem 2.3 and Corollary 2.4; in Section 10, we prove Theorems 2.5 and 2.6. In Section 11, we compare our proof to the proof in [3] of the propagation of regularity of the boundary of an aggregation patch.

### 3. Notation, conventions, and definitions

We define

\[ M_{m \times n}(\mathbb{R}) = \text{the space of all } m \times n \text{ real matrices}, \]

\[ M_j^i = \text{the element at the } i\text{-th row, } j\text{-th column of } M \in M_{m \times n}(\mathbb{R}), \]

\[ M_j = \text{the } j\text{-th column of } M \in M_{m \times n}(\mathbb{R}), \]
\[ M \cdot N = \sum_{i,j} M^i_j N^j_i = \sum_j M_j \cdot N_j \text{ for all } M, N \in M_{m \times n}(\mathbb{R}), \]

where repeated indices appearing in upper/lower index pairs are summed over, but no summation occurs if the indices are both upper or both lower.

We write \(|v|\) for the Euclidean norm of \(v = (v_1, v_2, \ldots, v^d)\), \(|v| = (v_1)^2 + (v_2)^2 + \cdots (v^d)^2\). For \(M \in M_{m \times n}(\mathbb{R})\), we use the operator norm,

\[
|M| := \max_{|v|=1} |Mv|. \tag{3.1}
\]

For \(M \in M_{d \times d}(\mathbb{R})\), let \(\text{cof} M\) be its cofactor matrix; thus, \((\text{cof} M)^i_j = (-1)^{i+j}\) times the \((i, j)\)-minor of \(M\). Form \(M \in M_{d \times d}(\mathbb{R})\) by letting its first \(d-1\) columns be \(Y_1, \ldots, Y_{d-1}\) and choosing its last column arbitrarily. We can define \(\wedge_{i<d} Y_i\) to be the last column of \((\text{cof} M)\), which we note is independent of the last column of \(M\); that is,

\[
\wedge_{i<d} Y_i := \text{cof} (Y_1 \ Y_2 \cdots Y_{d-1} \ Y_d). \tag{3.2}
\]

(We are really making a natural identification between the \((d-1)st\) exterior power, \(\Lambda^{d-1}\mathbb{R}^d\), which is one-dimensional, and \(\mathbb{R}^d\) itself.) In 2D and 3D,

\[
\wedge_{i<2} Y_i = Y_1^\top := (-Y_2^1, Y_1^1), \quad d = 2,
\]

\[
\wedge_{i<3} Y_i = Y_1 \times Y_2, \quad d = 3.
\]

**Definition 3.1** (Hölder spaces). Let \(U \subseteq \mathbb{R}^d\) be open. Then \(C^\alpha(U)\) is the space of all measurable functions for which

\[
\|f\|_{C^\alpha(U)} := \|f\|_{L^\infty(U)} + \|f\|_{C^\alpha(U)} < \infty, \quad \|f\|_{C^\alpha(U)} := \sup_{x,y \in U, x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.
\]

Note that

\[
\|f \circ g\|_{C^\alpha} \leq \|f\|_{C^\alpha} \|\nabla g\|_{L^\infty}, \quad \|f \circ g\|_{C^\alpha} \leq \|f\|_{L^\infty} + \|f\|_{C^\alpha} \|\nabla g\|_{L^\infty},
\]

\[
\|f \circ g\|_{C^\alpha} \leq \|f\|_{C^\alpha} \|g\|_{C^\alpha}, \quad \|1/f\|_{C^\alpha} \leq \frac{\|f\|_{C^\alpha}}{\inf|f|^\alpha}, \tag{3.3}
\]

The last inequality uses the homogeneous Lipschitz norm; if \(f\) is in \(C^1\) then \(\|f\|_{lip} = \|f\|_{L^\infty}\).

**Definition 3.2** (Radial cutoff functions). We make an arbitrary, but fixed, choice of a radially symmetric function \(a \in C_\infty^\infty(\mathbb{R}^d)\) taking values in \([0,1]\) with \(a = 1\) on \(B_1(0)\) and \(a = 0\) on \(B_2(0)^\circ\) for \(r > 0\), we define the rescaled cutoff function, \(a_r(x) = a(x/r)\), and for \(r, h > 0\) we define

\[
\mu_{r,h} = a_r(1 - a_h).
\]

When using the cutoff function \(\mu_{r,h}\), we will be fixing \(r\) while taking \(h \to 0\), in which case we can safely assume that \(h\) is sufficiently smaller than \(r\) so that \(\mu_{r,h}\) vanishes outside of \((h,2r)\) and equals 1 identically on \((2h,r)\). It will then follow that

\[
\begin{cases}
\langle \nabla \mu_{r,h} \rangle \leq C h^{-1} \leq C |x|^{-1} \text{ for } |x| \in (h, 2h), \\
|\nabla \mu_{r,h} | \leq C r^{-1} \leq C |x|^{-1} \text{ for } |x| \in (r, 2r), \\
\nabla \mu_{r,h} \equiv 0 \text{ elsewhere.}
\end{cases} \tag{3.4}
\]

Hence, we also have \(\langle \nabla \mu_{r,h} \rangle \leq C |x|^{-1}\) everywhere.
Lemma 3.3. Let \( r \in (0, 1) \). For all \( f \in C^\alpha(\mathbb{R}^d), \ g \in L^\infty(\mathbb{R}^d) \), we have
\[
\left| \int \nabla [\mu_{rh} \nabla \Phi](x-y)(f(x)-f(y))g(y) \, dy \right| \leq C \alpha^{-1} \|f\|_{C^\alpha} \|g\|_{L^\infty} r^\alpha.
\] (\( \Phi \) is the fundamental solution of the Laplacian as in (1.2).) For all \( f \in L^\infty(\mathbb{R}^d) \), we have
\[
\left| \int (\mu_{rh} \nabla \Phi)(x-y)f(y) \, dy \right| = \left| \int (\mu_{rh} \nabla \Phi)(x-y)(f(y)-f(x)) \, dy \right| \leq C \alpha^{-1} \|f\|_{L^\infty} r^\alpha.
\]

Proof. See Lemma 3.4 of [1]. (In [1], a negative Hölder space norm on \( f \) was used in place of \( L^\infty \) in the bound in (3.5), so the bound here, sufficient for our purposes, is not optimal.)

Definition 3.4. \( LL = LL(\mathbb{R}^d) \) is the space of bounded log-Lipschitz vector fields with
\[
\|g\|_{LL} := \|g\|_{L^\infty} + \sup \left\{ \frac{|g(x) - g(y)|}{-|x-y| \log |x-y|} : x, y \in \mathbb{R}^d, 0 < |x-y| \leq e^{-1} \right\}.
\]

4. SUFFICIENT FAMILIES

We adapt the concept of a sufficient family as defined in [8, 11] to better suit the manner in which we prepare the initial data, by organizing the vector fields into frames.

Definition 4.1. Let \( \mathbf{Y} = (Y_1, \ldots, Y_d) \) be an ordered set of \( C^\alpha \) vector fields on \( \mathbb{R}^d \) with \( Y_d = \wedge_{j<d} Y_j \) (see Section 3) and where \( \text{div} \, Y_1, \ldots, Y_{d-1} \in C^\alpha(\mathbb{R}^d) \). We call \( \mathbf{Y} \) a frame.

Let
\[
\mathcal{Y} = (Y^{(j)})_{j \in \Lambda} = ((Y^{(j)}_1, \ldots, Y^{(j)}_d))_{j \in \Lambda},
\] (4.1)
be a family of frames indexed by \( \Lambda \), which in our applications will be finite.

For any function \( f \) on vector fields (such as \( \text{div} \) or the identity map), define
\[
f(\mathcal{Y}) := ((f(Y^{(j)}_1), \ldots, f(Y^{(j)}_{d-1})))_{j \in \Lambda},
\] (4.2)
and define, for any Banach space \( X \),
\[
\|f(\mathcal{Y})\|_X := \sup_{j \in \Lambda, k < d} \|f(Y^{(j)}_k)\|_X.
\]

Note that we exclude the final vector in each frame. If \( \|f(\mathcal{Y})\|_X < \infty \) we say that \( f(\mathcal{Y}) \in X \).

Define
\[
I(\mathcal{Y}) := \inf_{x \in \mathbb{R}^d} \sup_{j \in \Lambda} |Y^{(j)}_d(x)|.
\] (4.3)

Definition 4.2. With \( \mathcal{Y} \) as above, we say that \( \mathcal{Y} \) is a sufficient family if
\[
\mathcal{Y} \in C^\alpha(\mathbb{R}^d), \ \ \text{div} \, \mathcal{Y} \in L^\infty(\mathbb{R}^d), \ \ \text{and} \ I(\mathcal{Y}) > 0.
\]

Let
\[
\mathcal{Y}_0 = (Y_0^{(j)})_{j \in \Lambda} = ((Y_{0,1}^{(j)}, \ldots, Y_{0,d}^{(j)}))_{j \in \Lambda}
\]
be a sufficient family. For any \( Y_{0,k}^{(j)}, k < d \), define its pushforward \( Y_k^{(j)}(t) \) (see Section 8 for more on pushforwards) by
\[
Y_k^{(j)}(t, \eta(t, x)) := (Y_{0,k}^{(j)}(x) \cdot \nabla) \eta(t, x).
\] (4.4)
(We assume here that \( \nabla \eta \in L^\infty((0,T) \times \mathbb{R}^d) \).) We then let \( Y^{(j)}_d(t) = \wedge_{k<d} Y^{(j)}_k(t) \) and define the pushforward \( \mathcal{Y}(t) \) of \( Y_0 \) as
\[
\mathcal{Y}(t) = (Y^{(j)}(t))_{j \in \Lambda} = ((Y^{(j)}_1(t), \ldots, Y^{(j)}_d(t)))_{j \in \Lambda}.
\]

5. **Serfati’s linear algebra lemma**

Lemma 5.1 is a version of a linear algebra lemma due to Serfati, appearing in various forms in [18, 19, 20]. The version here is a minor refinement of the version appearing in [1].

**Lemma 5.1.** For any symmetric \( B \in M_{d \times d}(\mathbb{R}) \), \( d \geq 1 \), we have
\[
|B| \leq \frac{P(Y_1, \ldots, Y_{d-1})}{|\wedge_{i<d} Y_i|^4} \sum_{i=1}^{d-1} |BY_i| + |\text{tr} B|,
\]
where \( Y_1, \ldots, Y_{d-1} \) are any linearly independent vectors in \( \mathbb{R}^d \) and \( P \) is a homogeneous polynomial in \( Y_1, \ldots, Y_{d-1} \) of degree \( n_d := 4d - 5 \).

**Proof.** Setting \( Y_d = \wedge_{k<d} Y_k \) as in our definition of a frame, form the matrix \( M \) by column as
\[
M := \begin{bmatrix} Y_1 & \cdots & Y_{d-1} & Y_d \end{bmatrix}.
\]
Note that \( Y_d \) is the last column of \( M \), the cofactor matrix of \( M \) (as in our definition of the wedge product itself in Section 1). Expanding about the last column of \( M \), we see that
\[
|Y_d|^2 = \det M \neq 0, \quad \because \text{we assumed that } Y_1, \ldots, Y_{d-1} \text{ are linearly independent.}
\]

Now,
\[
MM^T = MM^T = \det M \text{I},
\]
from which it follows that
\[
B = \frac{M}{(\det M)^2} A M^T, \quad A := M^T B M.
\]

Then, noting that \( A^d_j = Y_i \cdot BY_j \), we can write,
\[
A = \begin{pmatrix}
Y_1 \cdot BY_1 & \cdots & Y_1 \cdot BY_d \\
\vdots & \ddots & \vdots \\
Y_{d-1} \cdot BY_1 & \cdots & Y_{d-1} \cdot BY_d \\
Y_d \cdot BY_1 & \cdots & Y_d \cdot BY_d
\end{pmatrix}.
\]

Because \( B \) is symmetric, so is \( A \). Hence, we can replace \( Y_j \cdot BY_d \) in the final column with \( Y_d \cdot BY_j \), eliminating \( BY_d \) from all but \( A_d^d = Y_d \cdot BY_d \). For \( A_d^d \), we calculate,
\[
A_d^d = Y_d \cdot BY_d = M_d \cdot BY_d = \sum_{i=1}^{d} M_i \cdot BY_i - \sum_{i=1}^{d-1} M_i \cdot BY_i,
\]
where we used that \( M_d = Y_d \). But,
\[
\sum_{i=1}^{d} M_i \cdot BY_i = \sum_{i=1}^{d} (M^T BM)^i_i = \text{tr}(M^T BM) = \text{tr}(MM^T B) = \det M \text{ tr } B,
\]
so

\[ Y_d \cdot B Y_d = \det M \, \text{tr} \, B - \sum_{i=1}^{d-1} M_i \cdot B Y_i. \]

We conclude that

\[ A = E + \det M(\text{tr} \, B) D, \]

where

\[
E := \begin{pmatrix}
Y_1 \cdot B Y_1 & \cdots & Y_1 \cdot (BY)_{d-1} & Y_d \cdot B Y_1 \\
\vdots & \ddots & \vdots & \vdots \\
Y_{d-1} \cdot B Y_1 & \cdots & Y_{d-1} \cdot (BY)_{d-1} & Y_d \cdot B Y_{d-1} \\
Y_d \cdot B Y_1 & \cdots & Y_d \cdot (BY)_{d-1} & -\sum_{i=1}^{d-1} M_i \cdot B Y_i
\end{pmatrix}, \quad D := \begin{pmatrix} 0 & \cdots & 0 \\
& \ddots & \vdots & \vdots \\
& \vdots & \ddots & \vdots \\
& & & 0 \end{pmatrix}.
\]

Now,

\[
(MDM^T)^i_j = M^i_k D^k_l M^j_l = M^i_d M^j_d = Y_d^i Y_d^j,
\]

or,

\[ MDM^T = |Y_d|^2 (\text{tr} \, B) Y_d \otimes Y_d. \]

Hence,

\[
\frac{\det M(\text{tr} \, B) M}{|Y_d|^4 D M^T} \leq (\text{tr} \, B) \frac{|Y_d \otimes Y_d|}{|Y_d|^2} \leq \text{tr} \, B.
\]

As can be easily seen, the square of the operator norm of a square matrix is bounded above by the sum of the squares of the norms of the columns or of the rows. From this, (5.2), the bound above, and the form in which we have expressed \( E \), (5.1) follows. The degree and the homogeneity of \( P \) follow from scaling, or from summing the degrees of the factors making it up. \( \square \)

**Lemma 5.2.** Let \( U \) be an open subset of \( \mathbb{R}^d \), \( d \geq 2 \). Assume that \((Y_1, \ldots, Y_d)\) is a frame on \( U \) (so \( Y_d = \wedge_{i<d} Y_i \)) with \( \|Y_d\|_{C^\alpha} \geq C_0 > 0 \). For any symmetric \( B \in M_{d \times d}(\mathbb{R}) \), we have

\[ \|B\|_{C^\alpha} \leq C_0^{-8} Q(\|Y_1\|_{C^\alpha}, \ldots, \|Y_{d-1}\|_{C^\alpha}, \|BY_1\|_{C^\alpha}, \ldots, \|BY_{d-1}\|_{C^\alpha}) + \|\text{tr} \, B\|_{C^\alpha}, \]

for some polynomial, \( Q \), the \( C^\alpha \) norms being over \( U \).

**Proof.** The result follows from applying (3.2) to the decomposition of \( B \) given in the proof of Lemma 5.1. \( \square \)

6. **Lemmas involving the velocity gradient**

Throughout this section, we assume that

\[ \rho \in L^1 \cap L^\infty(\mathbb{R}^d), \quad v = -\nabla \Phi \ast \rho. \]

Proposition 6.1 is the extension of (7.12) of [3] to a general \( \rho \).

**Proposition 6.1.** We have,

\[
\nabla v(x) = -\frac{\rho(x)}{d} I - \text{p.v.} \int \nabla \nabla \Phi(x - y) \rho(y) \, dy.
\]
Proof. Let $H\Phi$ be the distributional Hessian matrix of $\Phi$. Then,

$$H\Phi = \text{p.v.} \nabla \nabla \Phi + \frac{1}{d} \delta_0 I,$$

where $\delta_0$ is the Dirac delta function at the origin. Using that $v = -\nabla \Phi * \rho$ completes the proof. \qed

The next proposition is the analog of Proposition 4.2 of [1].

**Proposition 6.2.** Let $Y$ be a vector field in $C^a(\mathbb{R}^d)$. Then

$$\text{p.v.} \int \nabla \nabla \Phi(x - y) Y(y) \rho(y) \, dy = -\nabla \Phi * \text{div}(\rho Y)(x) - \frac{\rho(x) Y(x)}{d}.$$

**Proof.** The proof is fairly standard, so we give only an outline. We assume first that $\rho \in C^\infty(\mathbb{R}^d)$. Noting that

$$[\nabla \nabla \Phi(x - y) Y(y)]^j = (\nabla \nabla \Phi)^{jk}(x - y) Y^k(y) = \partial_k \partial_j \Phi(x - y) Y^k(y) = \nabla \partial_j \Phi(x - y) \cdot Y(y),$$

integration by parts gives

$$\left[ \text{p.v.} \int \nabla \nabla \Phi(x - y) Y(y) \rho(y) \, dy \right]^j = \text{p.v.} \int \nabla \partial_j \Phi(x - y) \cdot Y(y) \rho(y) \, dy$$

$$= -\lim_{\delta \to 0} \int_{B_\delta^C(x)} \partial_j \Phi(x - y) \text{div}(\rho Y)(y) \, dy + \lim_{\delta \to 0} \int_{\partial B_\delta(x)} \partial_j \Phi(x - y) \rho(y) (Y(y) \cdot n) \, dy,$$

where $n$ is the unit outer vector on $\partial B_\delta(x)$. The area integral converges in the limit to $\partial_j \Phi * \text{div}(\rho Y)$ since $\nabla \Phi$ is locally integrable. For the boundary integral, one replaces $Y(y)$ by $Y(x)$, in which case the boundary integral converges to $-\rho(x)Y^j(x)/d$. Using that $|Y(x) - Y(y)| \leq \|Y\|_{C^a} |x - y|^\alpha$, the error in replacing $Y(y)$ by $Y(x)$ is easily seen to vanish as $\delta \to 0$. The result then follows by a density argument. \qed

**Corollary 6.3.** Let $Y \in C^a(\mathbb{R}^d)$. Then

$$Y(x) \cdot \nabla v(x) = -\text{p.v.} \int \nabla \nabla \Phi(x - y) [Y(x) - Y(y)] \rho(y) \, dy + \nabla \Phi * \text{div}(\rho Y)(x).$$

Moreover,

$$\left\| \text{p.v.} \int \nabla \nabla \Phi(x - y) [Y(x) - Y(y)] \rho(y) \, dy \right\|_{C^a} \leq CV(\rho) \|Y\|_{\dot{C}^a},$$

$$\|Y \cdot \nabla v(t)\|_{C^a} \leq CV(\rho) \|Y\|_{\dot{C}^a} + \|\text{div}(\rho Y)\|_{L^\infty},$$

where

$$V(\rho) := \|\rho\|_{L^\infty} + \left\| \text{p.v.} \int \nabla \nabla \Phi(\cdot - y) \rho(y) \, dy \right\|_{L^\infty} \quad (6.1)$$

**Proof.** The expression for $Y(x) \cdot \nabla v(x)$ follows from comparing the expressions in Proposition 6.1 and Proposition 6.2. The $C^a$-bound on the integral follows from fairly standard singular integral estimates (for instance, by applying Lemma 3.3 of [1] with the kernel $\rho(y)\nabla \nabla \Phi(x - y) [Y(x) - Y(y)]$.) For the bound on the convolution $\nabla \Phi * \text{div}(\rho Y)$, we first note

$$\|\nabla \Phi * \text{div}(\rho Y)\|_{C^a} \leq C \|\nabla \Phi * \text{div}(\rho Y)\|_{LL}$$

because

$$\frac{-|x - y| \log |x - y|}{|x - y|^\alpha} \leq C$$
when $|x - y| \leq e^{-1}$. We then apply the classical estimate $\|\nabla \Phi \ast f\|_{LL} \leq C \|f\|_{L^\infty}$, which holds for all $d \geq 2$ (see [17, Lemma 8.1] for a 2D proof), to obtain

$$\|\nabla \Phi \ast \text{div}(\rho Y)\|_{LL} \leq C \|\text{div}(\rho Y)\|_{L^\infty}.$$ 

The estimate $\|\nabla \Phi \ast f\|_{LL} \leq C \|f\|_{L^\infty}$ will also be used later in the proof of Theorem 2.3 in Section 9.2, (iii).

Corollary 6.3 is the extension to a general $\rho$ of Lemma 7.3 of [3]. We note that the term $\nabla \Phi \ast \text{div}(\rho Y)(x)$ appearing here is cancelled in the setting of [3] by the additional term $-\rho Y$ appearing in (7.6) of [3].

**Remark 6.4.** For any $i, j$,

$$(\nabla v)^i_j = \partial_j v^i = -\partial_j (\partial_i \Phi \ast \rho) = -(\partial_j \partial_i \Phi) \ast \rho = -(\partial_i \partial_j \Phi) \ast \rho = (\nabla v)^j_i,$$

so $\nabla v$ is symmetric, a fact will use in the following sections.

7. Preparing the initial data

In Proposition 7.1 we prepare the initial data and construct a sufficient family in such a way as to allow us to adapt the machinery created by Chemin in [7, 8] to obtain the propagation of striated regularity.

**Proposition 7.1.** Let $\rho_0$ be a suitable initial density, as in Definition 2.2, and let $\psi_j$ and $U_j$ be as in that definition. There exists a sufficient family $\mathcal{Y}_0$ and a sequence $(\rho_{0,n})_{n \in \mathbb{N}}$ of functions in $C^{1,\alpha}(\mathbb{R}^d)$, such that:

(i) Each $\rho_{0,n}$ is supported in a fixed compact $K \subseteq \mathbb{R}^d$;

(ii) $\rho_{0,n} \to \rho_0$ in $L^p(\mathbb{R}^d)$ for all $p \in [1, \infty)$;

(iii) $\|\rho_{0,n}\|_{L^\infty} \leq \|\rho_0\|_{L^\infty}$ for all $n \in \mathbb{N};$

(iv) $\text{div} \mathcal{Y}_0 = 0$;

(v) $\|\mathcal{Y}_0 \cdot \nabla \rho_{0,n}\|_{L^\infty} \leq C$ for all $n \in \mathbb{N}$.

The constant in (v) depends upon $\|\rho_0\|_{L^\infty}$ and $\psi_j$.

**Remark.** Recall that in evaluating a function, such as $\text{div}$, on a frame as in (4.2), we evaluate the function on all the vector fields in the frame except for the final one.

**Proof.** Because we deal only with initial data in this proof, we drop the time zero subscripts, writing $\rho$ and $\mathcal{Y}$, for instance, for $\rho_0$ and $\mathcal{Y}_0$. Recall from Definition 2.2 that $U_j$ is the interior of $\text{supp} \psi_j$.

Since $U_j$ is connected, $\psi_j(U_j) = (a_j, b_j)$ for some $-\infty < a_j < b_j < \infty$. Because $\psi_j$ is level-set-compatible with $\rho$ on $U_j$ as in Definition 2.1, $\rho \circ \psi_j^{-1}: (a_j, b_j) \to \mathbb{R}$ is well-defined: for any $r \in (a_j, b_j)$, $\rho \circ \psi_j^{-1}(r) = \rho(x)$, where $x$ is any point in $U_j$ for which $\psi_j(x) = r$. For any $n \in \mathbb{N}$, we can then define

$$\rho_{j,n}: U_j \to \mathbb{R}, \quad \rho_{j,n} = (\mu_{1/n} \ast \mathcal{E}_0(\rho \circ \psi_j^{-1})) \circ \psi_j,$$

where $\mathcal{E}_0$ is extension by zero from the domain $(a_j, b_j)$ to all of $\mathbb{R}$, and $(\mu_\varepsilon)_{\varepsilon \geq 1}$ is a family of 1D mollifiers with $\mu_\varepsilon(\cdot) = \varepsilon^{-d} \mu(\cdot/\varepsilon)$ and $\mu_1$ supported is on $(-1, 1)$. Because $\mu_{1/n} \ast \mathcal{E}_0(\rho \circ \psi_j^{-1}) \in C^\infty(\mathbb{R})$ and $\psi_j \in C^{1,\alpha}(\mathbb{R}^d)$, we see that $\rho_{j,n} \in C^{1,\alpha}(\mathbb{R}^d)$.

Any level set of $\psi_j$ maps to a level set of $\rho_{j,n}$, so $\psi_j$ is level-set compatible with $\rho_{j,n}$ on $U_j$. 

□
Let \( r_0 \) be as in Definition 2.2 and define

\[
U'_j = \{x \in U_j : \operatorname{dist}(x, \partial U_j) > r_0/2\}.
\]

(Note that \( U'_j \) could be empty, though only if \( U_j \subseteq V \), in which case we disregard \( U_j \) entirely.)

Construct a partition of unity \( (\varphi_j)_{j=1}^{N+1} \) with the property that for all \( j \leq N \), \( \varphi_j \equiv 1 \) on \( U'_j \) and \( \varphi_j \equiv 0 \) on \( \mathbb{R}^d \setminus U_j \). Then let \( \varphi_{N+1} = 1 - \sum_{j \leq N} \varphi_j \). Note that at each point in its support, \( \varphi_{N+1} = 1 - \varphi_j \) for one \( j \).

For any \( n \in \mathbb{N} \), define \( \rho_n \) by

\[
\rho_n := \sum_{j=1}^{N} \varphi_j \rho_{j,n} + \varphi_{N+1} \rho.
\]

Noting that \( \rho \in C^{1,\alpha}(V) \) and \( \operatorname{supp} \varphi_{N+1} \subseteq V \), we see that \( \rho_n \in C^{1,\alpha}(\mathbb{R}^d) \), and the \( \rho_n \) are supported in a common compact \( K \subseteq \mathbb{R}^d \) since \( \rho \) is compactly supported, giving (i). Basic properties of (1D) mollification give (ii) and (iii).

It remains to construct a sufficient family \( \mathcal{Y} \) satisfying (iv) and (v). In 2D, we need only \( N + 1 \) frames. We define the \( j \)th frame, \( Y^{(j)} = (Y_1^{(j)}, Y_2^{(j)}) \), by choosing

\[
Y_1^{(j)} = \nabla^{-1} \psi_j, \quad Y_2^{(j)} = \nabla \psi_j,
\]

noting that \( \operatorname{div} Y_1^{(j)} = 0 \), giving (iv). For (v), \( \psi_j \) is level-set compatible with \( \rho_{j,n} \) on \( U_j \), so \( Y_1^{(j)} \cdot \nabla \rho_{j,n} = 0 \). Also, \( Y_1^{(j)} \) vanishes on \( U_k \) for \( k \neq j \). Hence,

\[
Y_1^{(j)} \cdot \nabla \rho_n = \sum_{k=1}^{N} \left( \varphi_k Y_1^{(j)} \cdot \nabla \rho_{k,n} + \rho_{k,n} Y_1^{(j)} \cdot \nabla \varphi_k \right) + \varphi_{N+1} Y_1^{(j)} \cdot \nabla \rho + \rho Y_1^{(j)} \cdot \nabla \varphi_{N+1}
\]

\[
= \rho_{j,n} Y_1^{(j)} \cdot \nabla \varphi_j + \varphi_{N+1} Y_1^{(j)} \cdot \nabla \rho + \rho Y_1^{(j)} \cdot \nabla \varphi_{N+1}
\]

\[
= -\rho_{j,n} Y_1^{(j)} \cdot \nabla \varphi_{N+1} + \varphi_{N+1} Y_1^{(j)} \cdot \nabla \rho + \rho Y_1^{(j)} \cdot \nabla \varphi_{N+1}.
\]

Since also \( \rho \) is in \( C^{1,\alpha}(\operatorname{supp} \varphi_{N+1}) \), we see that each of the terms above is bounded in \( L^\infty \), from which (v) follows.

Finally, since \( \rho \) is \( C^{1,\alpha} \) on \( U_{N+1} \), we can set \( Y_{1}^{(N+1)} = \nabla^{-1}(\varphi_{N+1} e_1) \), \( Y_{2}^{(N+1)} = (Y^{(N+1)}_1) \perp \), and we can see that the family of frames, \( \mathcal{Y} = (Y^{(1)}, \ldots, Y^{(N+1)}) \), is sufficient.

The argument for \( d \geq 3 \) is similar to that for 2D, though now we need multiple frames to “cover” each \( U_j \)—these frames are given by applying Lemma 7.3, below, with \( W = U_j \), \( W' = U'_j \), and \( \psi = \psi_j \) to provide a family of frames \( \mathcal{Z}_j \). The full family of frames is obtained as the union, \( \mathcal{Y} = \bigcup \mathcal{Z}_1 \cup \cdots \cup \mathcal{Z}_N \) with one final frame added to cover \( U_{N+1} \) by applying Lemma 7.3, again with \( W = V \), \( W' = V \setminus \bigcup j U'_j \), and \( \psi = \varphi_{N+1} x_1 \).

It is shown by Foote in [13] that for a bounded domain \( \Omega \subseteq \mathbb{R}^d \), if \( \partial \Omega \) is \( C^k \) for \( k \geq 2 \) then there is a tubular neighborhood \( U \) of the boundary in the sense that every point in \( U \) has a unique closest point on \( \partial \Omega \). Moreover, although the projection map \( P : U \to \partial \Omega \) to this closest point is only \( C^{k-1} \), the signed distance function, \( \delta(x) = \pm \operatorname{dist}(x, \partial \Omega) \), chosen positive (negative) if \( x \) is outside (inside) is \( C^k \) on \( U \). The very simple and clean proofs in [13] extend, with almost no change, to a \( C^{1,\alpha} \) boundary, from which we can conclude that \( \delta \) is \( C^{1,\alpha} \) and \( P \) is \( C^\alpha \) on \( U \). Using this \( \delta \), it is not hard to apply Proposition 7.1 to obtain the following:
Corollary 7.2. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with $C^{1,\alpha}$ boundary and a finite number of connected components. Then $\rho_0 := c_1\Omega$, where $c$ is a constant, is a suitable initial density as in Definition 2.2.

We used the following lemma in the proof of Proposition 7.1.

Lemma 7.3. Let $W$ and $W' \subseteq W$ be nonempty open subsets of $\mathbb{R}^d$. Let $\psi \in C^{1,\alpha}(\mathbb{R}^d)$ with $|\nabla \psi| \geq C_0 > 0$ on $W'$. Then there exists a family of frames,

$$Z = ((Z_1^{(k)}, \ldots, Z_d^{(k)}))_{k=1}^d,$$

with each $Z_j^{(k)}$ in $C^\alpha(W)$ and for which $\text{div} \, Z_j^{(k)} \cdot \nabla \psi = 0$ for all $j < d$. Moreover, $I_{|W'}(Z) > 0$, where $I_{|W'}$ is defined as in (4.3), but restricted to $W'$.

Proof. To illustrate the approach, we first show how to choose $Z$ when $d = 3$. We set

$$
Z_1^{(1)} = (-\partial_2 \psi, \partial_1 \psi, 0), \quad Z_2^{(1)} = (-\partial_3 \psi, 0, \partial_1 \psi),
$$

$$
Z_2^{(2)} = (\partial_2 \psi, -\partial_1 \psi, 0), \quad Z_2^{(3)} = (0, -\partial_3 \psi, \partial_2 \psi),
$$

$$
Z_3^{(3)} = (\partial_3 \psi, 0, -\partial_1 \psi), \quad Z_2^{(3)} = (0, 0, -\partial_3 \psi, -\partial_2 \psi).
$$

We also set $Z_3^{(k)} = Z_1^{(k)} \wedge Z_2^{(k)}$ for each $k$. Then, $\text{div} \, Z_j^{(k)} = 0$ and $Z_j^{(k)} \cdot \nabla \psi = 0$ for all $j < 3$, $k \leq 3$. Moreover, for any $x_0 \in W'$, at least one component of $\nabla \psi(x_0)$, $\partial_k \psi(x_0)$, must have $|\partial_k \psi(x_0)| \geq C_0/\sqrt{3}$. Since $\nabla \psi$ is continuous, $\partial_k \psi \geq C_0/2\sqrt{3}$ on some open set containing $x_0$. Moreover, it is clear that $Z_1^{(k)}$ and $Z_2^{(k)}$ are linearly independent, and that locally to $x_0$, $|Z_3^{(k)}| = |\nabla \psi|^2 \geq C_0^2$ is bounded away from zero. This implies that $I_{|W'}(Z) > 0$.

We next take $d = 4$. We set

$$
Z_1^{(1)} = (-\partial_2 \psi, \partial_1 \psi, 0, 0), \quad Z_2^{(1)} = (-\partial_3 \psi, 0, \partial_1 \psi, 0), \quad Z_3^{(1)} = (-\partial_4 \psi, 0, 0, \partial_1 \psi),
$$

$$
Z_2^{(2)} = (\partial_2 \psi, -\partial_1 \psi, 0, 0), \quad Z_2^{(3)} = (0, -\partial_3 \psi, \partial_2 \psi, 0), \quad Z_3^{(2)} = (0, -\partial_4 \psi, 0, \partial_2 \psi),
$$

$$
Z_3^{(3)} = (\partial_3 \psi, 0, -\partial_1 \psi, 0), \quad Z_2^{(3)} = (0, \partial_3 \psi, -\partial_2 \psi, 0), \quad Z_3^{(3)} = (0, 0, -\partial_1 \psi, \partial_3 \psi),
$$

$$
Z_4^{(4)} = (\partial_4 \psi, 0, 0, -\partial_1 \psi), \quad Z_4^{(4)} = (0, 0, \partial_4 \psi, 0, -\partial_2 \psi), \quad Z_3^{(4)} = (0, 0, 0, \partial_4 \psi, -\partial_3 \psi).
$$

We also set $Z_4^{(k)} = Z_1^{(k)} \wedge Z_2^{(k)} \wedge Z_3^{(k)}$ for each $k$. We see that on all of $W$, $\text{div} \, Z_j^{(k)} = 0$ and $Z_j^{(k)} \cdot \nabla \psi = 0$ for all $j < 4$, $k \leq 4$. Moreover, for any $x_0 \in W'$, at least one component of $\nabla \psi(x_0)$, $\partial_k \psi(x_0)$, must have $|\partial_k \psi(x_0)| \geq C_0/2$. Since $\nabla \psi$ is continuous, $\partial_k \psi \geq C_0/4$ on some open set containing $x_0$. Moreover, it is clear that $Z_1^{(k)}$, $Z_2^{(k)}$, $Z_3^{(k)}$ are linearly independent, and that locally to $x_0$, $|Z_4^{(k)}| = |\nabla \psi|^2 \geq C_0^2$ is bounded away from zero. This implies that $I_{|W'}(Z) > 0$.

The extension to any $d \geq 2$ is then clear. \qed

8. Pushforwards and transport

We establish the basic transport equations for $Y$, $\text{div} \, Y$, and $Y \cdot \nabla \rho$ in Proposition 8.1, and use them (in part) to establish the bounds in Proposition 8.2.

Proposition 8.1. Assume that $\rho$ is a solution to (1.1, 1.4) for which the velocity, $v$, is at least Lipschitz continuous in space uniformly in time and that $\nabla \rho(t)$ is at least continuous in time and space. Let $\eta$ be the corresponding unique flow map for $v$ as in (1.6). Define the pushforward of the vector field $Y_0 \in C^\alpha(\mathbb{R}^d)$ by

$$Y(t, \eta(t, x)) = Y_0(x) \cdot \nabla \eta(t, x).$$

(8.1)
Then,

\[ \partial_t Y + v \cdot \nabla Y = Y \cdot \nabla v, \quad \text{(8.2)} \]
\[ \partial_t \operatorname{div} Y + v \cdot \nabla \operatorname{div} Y = -Y \cdot \nabla \rho, \quad \text{(8.3)} \]
\[ \partial_t (Y \cdot \nabla \rho) + v \cdot \nabla (Y \cdot \nabla \rho) = F'(\rho)Y \cdot \nabla \rho. \quad \text{(8.4)} \]

**Proof.** By our assumptions, \( \operatorname{div}(\rho Y) \) and \( Y \cdot \nabla \rho \) are continuous in time and space. The identity in (8.2) is classical. Taking the divergence of each side of (8.2), we obtain

\[ \partial_t \operatorname{div} Y + v \cdot \nabla \operatorname{div} Y = Y \cdot \nabla \operatorname{div} v = -Y \cdot \nabla \rho, \]

giving (8.3).

Using (1.7) and (8.2), we have

\[ \frac{d}{dt} (\rho Y)(t, \eta(t, x)) = \rho(t, \eta(t, x)) \frac{d}{dt} Y(t, \eta(t, x)) + Y(t, \eta(t, x)) \frac{d}{dt} \rho(t, \eta(t, x)) \]
\[ = \rho(t, \eta(t, x))(Y \cdot \nabla v)(t, \eta(t, x)) + F'(\rho(t, \eta(t, x))) Y(t, \eta(t, x)), \quad \text{(8.5)} \]

which we write more compactly as

\[ \partial_t (\rho Y) + v \cdot \nabla (\rho Y) = \rho Y \cdot \nabla v + F'(\rho)Y. \]

Noting that the derivation of (8.3) from (8.2) did not use the specific definition of \( Y \) in (8.1), the same argument applies with \( \rho Y \) in place of \( Y \). We have, however, the additional term on the right-hand side,

\[ \operatorname{div}(F'(\rho)Y) = Y \cdot \nabla (F'(\rho)) + F'(\rho) \operatorname{div} Y = F'(\rho)Y \cdot \nabla \rho + F'(\rho) \operatorname{div} Y, \]

so we conclude that

\[ \partial_t \operatorname{div}(\rho Y) + v \cdot \nabla \operatorname{div}(\rho Y) = -\rho Y \cdot \nabla \rho + F'(\rho)Y \cdot \nabla \rho + F'(\rho) \operatorname{div} Y. \quad \text{(8.6)} \]

Then, since \( \operatorname{div}(\rho Y) = Y \cdot \nabla \rho + \rho \operatorname{div} Y \), we have

\[ \partial_t \operatorname{div}(\rho Y) + v \cdot \nabla \operatorname{div}(\rho Y) \]
\[ = \partial_t (Y \cdot \nabla \rho) + \rho \partial_t \operatorname{div} Y + v \cdot \nabla \operatorname{div} Y + (\partial_t \rho + v \cdot \nabla \rho) \operatorname{div} Y + v \cdot \nabla (Y \cdot \nabla \rho) \]
\[ = \partial_t (Y \cdot \nabla \rho) + v \cdot \nabla (Y \cdot \nabla \rho) - \rho Y \cdot \nabla \rho + F'(\rho) \operatorname{div} Y, \]

where we used (1.1)\(_1 \) and (8.3). By (8.6) it follows that

\[ \partial_t (Y \cdot \nabla \rho) + v \cdot \nabla (Y \cdot \nabla \rho) = F'(\rho)Y \cdot \nabla \rho, \]

which is (8.4). \( \square \)

**Proposition 8.2.** Assume that \( \rho_0 \) is a suitable initial density as in Definition 2.2, let \( (\rho_n, n) \) be given by Proposition 7.1, let \( \rho_n \) be the corresponding \( C^1(0, T; C^1(\mathbb{R}^d)) \)-solutions to (1.1, 1.4), and let \( \mathcal{Y}_n \) be the pushforward of \( \mathcal{Y} \) by the flow map \( \eta_n \). If \( \operatorname{div} \mathcal{Y}_0 = Y_0 \cdot \nabla \rho_0, n = 0 \) on the open set \( W \) then \( \mathcal{Y}_n(t) = Y_n \cdot \nabla \rho_n, n = 0 \) on \( \eta_n(t, W) \) for all \( t \in [0, T] \). Moreover,

\[ \| \mathcal{Y}_n \cdot \nabla \rho_n(t) \|_{L^\infty} \leq \| Y_0 \cdot \nabla \rho_0 \|_{L^\infty} e^{C(T)t}, \]
\[ \| \operatorname{div} \mathcal{Y}_n \|_{L^\infty} \leq \| Y_0 \cdot \nabla \rho_0 \|_{L^\infty} e^{C(T)t}. \quad \text{(8.7)} \]

**Proof.** Fix \( x \in W \) and let \( Y_0 \) be one of the first \( d - 1 \) vector fields in any frame in \( \mathcal{Y}_0 \). Define

\[ g(t) := Y_n \cdot \nabla \rho_n(t, \eta_n(t, x)), \quad h(t) := \operatorname{div} Y_n(t, \eta_n(t, x)), \quad \overline{\rho}(t) := \rho_n(t, \eta_n(t, x)), \]
where \( Y_n = Y_0(t) \) is the pushforward of \( Y_0 \) by the flow map, \( \eta_n \). Then from (8.3) and (8.4) we see that \( g(t), h(t) \) solve the ODEs,

\[
\begin{align*}
\dot{g}(t) &= F'(\overline{\rho}(t))g(t), \quad g(0) = Y_0 \cdot \nabla \rho_0,t(x), \\
\dot{h}(t) &= -g(t), \quad h(0) = \text{div} \, Y_0(x).
\end{align*}
\]

Along flow lines, \( \rho \) is Lipschitz continuous (as we can see from (1.8)), so \( \overline{\rho} \) is locally Lipschitz. Also, since \( h(0) = 0 \) for all \( x \in \mathbb{R}^d \) (by Proposition 7.1) and \( g(0) = 0 \) for \( x \in W \), we see that \( Y \cdot \nabla \rho(t, x) = \text{div} \, Y(t, x) = 0 \) for \( x \in \eta(t, W) \).

Since we assumed that \( F' \) is locally Lipschitz, we can integrate the equations for \( g \) then \( h \) to obtain (after calculating that \( (\log g(t))' = F'(\overline{\rho}(t)) \) and using that \( h(0) = 0 \) on \( \mathbb{R}^d \)),

\[
g(t) = g(0) \exp \int_0^t F'(\overline{\rho}(s)) \, ds, \quad h(t) = -\int_0^t g(s) \, ds.
\]

Since \( \rho \in L^\infty([0, T] \times \mathbb{R}^2) \), \( |F'(\overline{\rho}(s))| \leq C(T) \). This gives the simple bounds,

\[
|g(t)| \leq e^{C(T)t} |g(0)|, \quad |h(t)| \leq te^{C(T)t} |g(0)|,
\]

from which, along with (v) of Proposition 7.1, (8.7) follows. \( \square \)

Proposition 8.3, which we will apply in the proof of Theorem 2.6, gives estimates on the propagation of local regularity of the density. Whereas in Proposition 8.2 we needed to work with regular solutions to make sense of the pushforward of \( Y_0 \) and to make sense of the term \( Y \cdot \nabla \rho \) appearing in the transport equations, in Proposition 8.3, we need only the regularity of a weak solution. Without, however, higher regularity of the flow map, such as that obtained in Theorem 2.3, the bounds will all be infinite.

**Proposition 8.3.** Let \( \rho \) be a weak Lagrangian solution to (1.1, 1.4) with initial density \( \rho_0 \), as in Definition 1.1. Then for any open set \( W \subseteq \mathbb{R}^d \),

\[
\begin{align*}
\|\rho(t)\|_{C^\alpha(\eta(t,W))} &\leq \|\rho_0\|_{C^\alpha(W)} \, e^{C(T)J(t)}, \\
\|
abla \rho(t)\|_{L^\infty(\eta(t,W))} &\leq \|
abla \rho_0\|_{L^\infty(W)} \, e^{C(T)+J(t)}, \\
\|
abla \rho(t)\|_{C^\alpha(\eta(t,W))} &\leq \|
abla \rho_0\|_{C^\alpha(W)} \, e^{C(T)+J(t)} \, \|
abla \eta^{-1}(t)\|_{C^\alpha(\eta(t,W))},
\end{align*}
\]

where \( J(t) := \int_0^t \|\nabla v(s)\|_{L^\infty} \, ds \) and \( C(T) \) depends continuously on \( F, \|\rho_0\|_{L^\infty} \), and \( T \).

*Proof.* Before proceeding further, observe that

\[
\|
abla \eta(t)\|_{L^\infty}, \quad \|
abla \eta^{-1}(t)\|_{L^\infty} \leq e^{J(t)}.
\]

The bound on the forward flow map follows directly from the integral form of (1.6),

\[
\eta(t, x) = x + \int_0^t v(s, \eta(s, x)) \, ds,
\]

while the bound on the inverse flow map is derived later in Lemma 8.4.

Let

\[
L(s, x) := \rho(s, \eta(s, x)).
\]

Then (1.8) can be expressed in the form,

\[
L(t, x) = \rho_0(x) + \int_0^t F(L(s, x)) \, ds.
\]
Applying Osgood’s lemma (see ([9, Lemma 5.2.1]), we see that
\[ \int_{|\rho_0(x)|}^{L(t,x)} \frac{dr}{|F(r)|} \leq t. \]

This gives the finite time, \( T^* \), up to which \( L(t,x) \) remains bounded by some constant \( C(T) \); this is the origin of (1.10). That is, \( \|L(t)\|_{L^\infty} = \|\rho(t)\|_{L^\infty} \leq C(T) \), and there exists a bounded \( E \subseteq (0,T) \times \mathbb{R}^d \), such that
\[ \text{image}(L) \subseteq E. \]

That \( E \) is bounded will be important when applying (3.2), which we do multiple times below.

Returning to (8.11), we can put the bound on \( \|L(t)\|_{L^\infty} \) in the form
\[ \|L(t)\|_{L^\infty} = \|\rho(t)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty} e^{C(T)t}, \tag{8.12} \]
though we have no effective method for calculating \( C(T) \), which also depends upon \( \rho_0 \), without an explicit choice of \( F \).

We now bound \( L(t) \) in \( C^\alpha(W) \) and \( C^{1+\alpha}(W) \). So assume \( x \in W \). Then from (8.11),
\[ \|L(t)\|_{C^\alpha(W)} \leq \|\rho_0\|_{C^\alpha(W)} + \int_0^t \|F'(L(s,x))\|_{L^\infty(E)} \|L(s)\|_{C^\alpha(W)} \, ds, \]
where we used (3.2). Thus, using also (8.12),
\[ \|L(t)\|_{C^\alpha(W)} \leq \|\rho_0\|_{C^\alpha(W)} e^{C(T)t} \]
follows from Grönwall’s lemma.

Similarly, applying \( \nabla \) to (8.11) gives
\[ \nabla L(t,x) = \nabla \rho_0(x) + \int_0^t F'(L(s,x)) \nabla L(s,x) \, ds. \tag{8.13} \]

Hence,
\[ \|\nabla L(t)\|_{L^\infty(W)} \leq \|\nabla \rho_0\|_{L^\infty(W)} + \int_0^t \|F'(L(s,x))\|_{L^\infty(W)} \|\nabla L(s)\|_{L^\infty(W)} \, ds \]
and Grönwall’s lemma then gives
\[ \|\nabla L(t)\|_{L^\infty(W)} \leq \|\nabla \rho_0\|_{L^\infty(W)} e^{Ct}. \tag{8.14} \]

We can also use (8.13) to bound the \( C^\alpha \) norm. First, observe that by (3.2),
\[ \|F'(L(s,x))\|_{C^\alpha(W)} = \|F'(L(s,x))\|_{L^\infty} + \|F'(L(s,x))\|_{C^\alpha} \]
\[ \leq C(T) + \|F'(L(s,x))\|_{lip(E)} \|L(s)\|_{C^\alpha(W)} \leq C(T), \]
by our earlier bound on \( \|L(t)\|_{C^\alpha(W)} \). Hence,
\[ \|\nabla L(t)\|_{C^\alpha(W)} \leq \|\nabla \rho_0\|_{C^\alpha(W)} + \int_0^t \|F'(L(s,x))\|_{C^\alpha(W)} \|\nabla L(s)\|_{C^\alpha(W)} \, ds \]
\[ \leq \|\nabla \rho_0\|_{C^\alpha(W)} + C(T) \int_0^t \|\nabla L(s)\|_{C^\alpha(W)} \, ds. \]
Then Grönwall’s lemma combined with (8.14) give
\[ \|\nabla L(t)\|_{C^\alpha(W)} \leq \|\nabla \rho_0\|_{C^\alpha(W)} e^{C(T)t}. \]
Then (3.2) also gives
\[ \|\rho(t)\|_{C^0(\eta(t,W))} = \|L(t,\eta^{-1}(t,\cdot))\|_{C^0(\eta(t,W))} \leq \|L(t)\|_{C^0(\eta(t,W))} \|\nabla\eta^{-1}(t)\|_{\ell^\infty} \]
\[ \leq \|\rho_0\|_{C^0(W)} e^{C(t)\tau + J(t)}. \]
And \(\nabla \rho(t) = \nabla (L(t,\eta^{-1}(t,x))) = (\nabla L)(t,\eta^{-1}(t,x)) \cdot \nabla \eta^{-1}(t,x)\). Thus,
\[ \|\nabla \rho(t)\|_{L^\infty(\eta(t,W))} \leq \|\nabla L(t)\|_{L^\infty(W)} \|\nabla \eta^{-1}(t)\|_{L^\infty(\eta(t,W))} \leq \|\nabla \rho_0\|_{L^\infty(W)} e^{C(t)\tau + J(t)}. \]
Because
\[ \|\nabla (L(t,\eta^{-1}(t,\cdot)))\|_{C^0(\eta(t,W))} \leq \|\nabla (L(t,\eta^{-1}(t,\cdot)))\|_{L^\infty(\eta(t,W))} + \|\nabla (L(t,\eta^{-1}(t,\cdot)))\|_{C^0(\eta(t,W))} \]
\[ \leq \|\nabla \rho(t)\|_{L^\infty(\eta(t,W))} + \|\nabla L(t)\|_{C^0(W)} \|\nabla \eta^{-1}(t)\|_{L^\infty}, \]
we also have (with the bounds above),
\[ \|\nabla \rho(t)\|_{C^0(\eta(t,W))} \leq \|\nabla (L(t,\eta^{-1}(t,\cdot)))\|_{C^0(W)} \|\nabla \eta^{-1}(t)\|_{C^0(\eta(t,W))} \]
\[ \leq \|\nabla \rho_0\|_{C^0(W)} e^{C(t)\tau + J(t)} \|\nabla \eta^{-1}(t)\|_{C^0(\eta(t,W))}, \]
from which (8.8) follows. \(\square\)

The following lemma gives estimates for the inverse flow map. As a general rule, the bounds on spatial (though not temporal) regularity of an inverse flow map are the same as those for the forward flow map, just more involved to derive for non-autonomous flows.

**Lemma 8.4.** Let \(\eta\) be the flow map on \([0, T] \times \mathbb{R}^d\) for a vector \(v\) with \(v, \nabla v \in L^\infty((0, T) \times \mathbb{R}^d)\). Then
\[ \|\nabla \eta^{-1}(t)\|_{L^\infty} \leq \exp \int_0^t \|\nabla v(\tau)\|_{L^\infty} \ d\tau. \] (8.15)
If further, \(v(t) \in C^{1,\alpha}(\eta(t,W))\) for some open \(W \subseteq \mathbb{R}^d\) with a uniform bound over \(t \in [0, T]\) then
\[ \|\nabla \eta^{-1}(t)\|_{C^0(\eta(t,W))} \leq e^{e^{C(T)t}}, \] (8.16)
where we have not been explicit about the constant \(C(T)\).

**Proof.** Suppose that a particle moving under the flow map is at position \(x\) at time \(t\). Let \(\mu(\tau; t, x)\) be the position of that same particle at time \(t - \tau\), where \(0 \leq \tau \leq t\). Then
\[ \eta^{-1}(t, x) = \mu(t; t, x), \quad x = \mu(0; t, x) \]
and
\[ \frac{d}{d\tau} \mu(\tau; t, x) = -v(t - \tau, \mu(\tau; t, x)). \]
By the fundamental theorem of calculus,
\[ \mu(s; t, x) - x = \int_0^s \frac{d}{d\tau} \mu(\tau; t, x) \ d\tau, \]
or,
\[ \mu(s; t, x) = x - \int_0^s v(t - \tau, \mu(\tau; t, x)) \ d\tau. \]
Taking the spatial gradient, we have
\[ \nabla \mu(s; t, x) = I - \int_0^s (\nabla v)(t - \tau, \mu(\tau; t, x)) \nabla \mu(\tau; t, x) \, d\tau. \] (8.17)
Hence,
\[ \| \nabla \mu(s; t, \cdot) \|_{L^\infty} \leq 1 + \int_0^s \| \nabla v(t - \tau) \|_{L^\infty} \| \nabla \mu(\tau; t, \cdot) \|_{L^\infty} \, d\tau. \]
It follows from Grönwall’s lemma that
\[ \| \nabla \mu(s; t, \cdot) \|_{L^\infty} \leq \exp \int_0^s \| \nabla v(t - \tau) \|_{L^\infty} \, d\tau \] (8.18)
for all \( s \in [0, t] \). Setting \( s = t \) gives (8.15).

Now we apply (3.2) to (8.17), with \( x \) restricted to lie in \( \eta(t, W) \), noting that \( \mu(\tau; t, x) \in \eta(t - \tau, W) \). This gives
\[ \| \nabla \mu(s; t, x) \|_{C^0(\eta(t-s, W))} \leq 1 + \int_0^s \| \nabla v(t - \tau) \|_{L^\infty} \| \nabla \mu(\tau; t, \cdot) \|_{C^0(\eta(t-\tau, W))} \, d\tau \]
\[ \leq 1 + \int_0^s \| \nabla v(t - \tau) \|_{C^0} \| \nabla \mu(\tau; t, \cdot) \|_{C^0} \| \nabla \mu(\tau; t, \cdot) \|_{C^0(\eta(t-\tau, W))} \, d\tau \]
\[ \leq 1 + C(T)e^{Ct} \int_0^s \| \nabla \mu(\tau; t, \cdot) \|_{C^0(\eta(t-\tau, W))} \, d\tau, \]
where we applied (8.18). Grönwall’s lemma gives
\[ \| \nabla \mu(s; t, x) \|_{C^0(\eta(t-s, W))} \leq \exp \left( C(T)e^{Ct} \right), \]
and setting \( s = t \) gives (8.16).

The following proposition will be used in the proof of Theorem 2.3 in Section 9.

**Proposition 8.5.** Let \( Y_0 \) be a frame pushed forward to \( Y \) by the flow map for the solution to (1.1, 1.4). Then
\[ \partial_t Y_d + v \cdot \nabla Y_d = -Y_d \cdot \nabla v - \rho Y_d. \]

**Proof.** Recall that be the definition of a frame, \( Y_d(t) = \wedge_{k < d} Y_k(t) \). A direct computation from our definition of the wedge product in Section 3 (using that \( \nabla v \) is symmetric) gives
\[ \partial_t Y_d + v \cdot \nabla Y_d = -Y_d \cdot \nabla v + (\text{div } v) Y_d. \]
The results follows since \( \text{div } v = -\rho \). (The special case when \( \text{div } v = 0 \) with \( (\nabla v)^T \) in place of \( \nabla v \) appears in the proof of Proposition 4.1 of [11].)

**Proposition 8.6.** Let \( \phi \) be \( \phi_0 \) (purely) transported by the flow, \( \partial_t \phi + v \cdot \nabla \phi = 0 \). Let \( Y \) be \( Y_0 \) pushed forward according to (4.4) and let \( W_0 = \nabla \phi_0 \) and \( W = \nabla \phi \). Assume that \( W_0 \cdot Y_0 = 0 \). Then, \( W \cdot Y = 0 \) for all time.

**Proof.** We have
\[ \partial_t \nabla \phi + \nabla (v \cdot \nabla \phi) = 0. \]
But
\[ (\nabla (v \cdot \nabla \phi))^i = \partial_i (v^j \partial_j \phi) = v^j \cdot \partial_j \partial_i \phi + \partial_i v^j \partial_j \phi = v \cdot \nabla (\nabla \phi)^i + (\nabla v)^T \nabla \phi) \]
\[ = (v \cdot \nabla \nabla \phi)^i + (\nabla v \nabla \phi)^i, \]
since $\nabla v$ is symmetric. Hence,

$$\partial_t \nabla \phi + v \cdot \nabla \nabla \phi = -\nabla v \nabla \phi,$$

or

$$\frac{d}{dt} (W(t, \eta(t, x))) = (\partial_t W + v \cdot \nabla W)(t, \eta(t, x)) = -(W \cdot \nabla v)(t, \eta(t, x)).$$

Assume that $Y$ is obtained from (4.4). Then, since,

$$\frac{d}{dt} (Y(t, \eta(t, x))) = (\partial_t Y + v \cdot \nabla Y)(t, \eta(t, x)) = (Y \cdot \nabla v)(t, \eta(t, x)),$$

we have

$$\frac{d}{dt} ((W \cdot Y)(t, \eta(t, x))) = W(t, \eta(t, x)) \cdot \frac{d}{dt} (Y(t, \eta(t, x))) + Y(t, \eta(t, x)) \cdot \frac{d}{dt} (W(t, \eta(t, x)))$$

$$= (W \cdot (Y \cdot \nabla v) - Y \cdot (W \cdot \nabla v))(t, \eta(t, x)).$$

But, for two vector fields, $Y$ and $Z$, we have

$$Y \cdot (Z \cdot \nabla v) - Z \cdot (Y \cdot \nabla v) = Y^i (Z^j \partial_j v^i) - Z^j (Y^i \partial_i v^j) = 0,$$

again using that $\nabla v$ is symmetric. Therefore,

$$\frac{d}{dt} (W \cdot Y)(t, \eta(t, x)) = 0.$$

We conclude that if $W_0 \cdot Y_0 = 0$ then $W \cdot Y = 0$ for all time. □

We will find Proposition 8.7 useful when changing variables using the flow map:

**Proposition 8.7.** Let $\rho$, $\eta$ be as in Proposition 8.1. Then $\det \nabla \eta(t, x)$, the Jacobian determinant of the map $x \rightarrow \eta(t, x)$ (which is positive), is bounded by

$$\exp \left(-\int_0^t |\rho(s, \eta(s, x))| \, ds \right) \leq \det \nabla \eta(t, x) \leq \exp \left(\int_0^t |\rho(s, \eta(s, x))| \, ds \right).$$

The same bound applies to $\det \nabla \eta^{-1}(t, x)$.

**Proof.** Fix $x \in \mathbb{R}^d$ and let $f(t) = \det \nabla \eta(t, x), \, \bar{\rho}(t) = \rho(t, \eta(t, x))$. Then (see, for instance, page 3 of [9])

$$f'(t) = \text{div} v(t, \eta(t, x)) f(t) = -\bar{\rho}(t) f(t).$$

It follows that $(\log f(t))' = -\bar{\rho}(t)$, so since also $f(0) = 1$,

$$\log f(t) = -\int_0^t \bar{\rho}(s) \, ds \implies f(t) = \exp \left(-\int_0^t \bar{\rho}(s) \, ds \right).$$

This gives the bounds on $\det \nabla \eta(t, x)$. Since $\eta(\cdot, x)$ is a diffeomorphism, the same bounds apply on its inverse. □
9. Propagation of striated regularity: proof of Theorem 2.3

It would be quite convenient if we could implement the proof of Theorem 2.3 using the weak solution provided by Theorem 1.2 directly. The problem is that to define the pushforward of $Y_0$, we need $v$ to at least be Lipschitz so that $\nabla \eta$ is defined (and bounded), whereas we only know a priori that the weak solution has a log-Lipschitz $v$. Beyond this, we need higher regularity than $v$ Lipschitz to justify the transport equations in (8.3) and (8.4). Hence, we will need to use a sequence of approximating regular solutions, $(\rho_n)$, and, as part of the proof, show that there is a uniform bound on $(\nabla v_n)$ in $L^\infty((0,T) \times \mathbb{R}^d)$ so that for the limiting weak solution, $\nabla v$ will lie in $L^\infty((0,T) \times \mathbb{R}^d)$.

This is much as in the proof of the propagation of striated regularity of the vorticity for the 2D Euler equations following any of the existing approaches in [7, 8, 20, 1]. A key difference arises, however, in the transport equations in (8.3) and (8.4): for the 2D Euler equations these are pure transport, whereas we must deal with a troublesome right-hand side. Also, proof that the sequence of approximating solutions converges to a weak solution to (1.1, 1.4) will require more work than for the Euler equations because the velocity fields are not divergence-free.

We will first establish in Proposition 9.1 a series of estimates for solutions to (1.1, 1.4) with the initial data regularized as in Proposition 7.1. These estimates will control $Y$ and $Y \cdot \nabla v$ in terms of the initial data and the quantity,

$$V(t) := \|\rho_0\|_{L^\infty} + \left\| \text{p. v.} \int_{\mathbb{R}^d} \nabla \nabla \Phi(x-y) \rho(y) \, dy \right\|_{L^\infty_x(\mathbb{R}^d)}.$$

The key part of the proof of Proposition 9.1 lies in establishing a useful bound on $V(t)$ itself so that we can ultimately close the estimates.

We give the proof of Proposition 9.1 in Section 9.1, using it to prove Theorem 2.3 in Section 9.2. In Section 9.3 we give the proof of Corollary 2.4.

9.1. Uniform estimates on approximating solutions. In this subsection we establish Proposition 9.1, a series of uniform estimates for the approximating regular solutions to (1.1, 1.4). The estimates in (9.1) through (9.4) are based primarily upon the gradient flow structure of the transport equations in Section 8. Obtaining the estimates in (9.4) and (9.5) will consume most of our effort.

Proposition 9.1. Let $(\rho_n)$ be the sequence of approximating $C^{1,\alpha}$-solutions to (1.1, 1.4) for the suitable initial density $\rho_0$, as in Proposition 8.2. Let $v_n$, $\eta_n$, and $Y_n$ the corresponding velocity field, flow map, and pushforward of $Y_0$. Define,

$$V_n(t) := \|\rho_0\|_{L^\infty} + \left\| \text{p. v.} \int_{\mathbb{R}^d} \nabla \nabla \Phi(x-y) \rho_n(y) \, dy \right\|_{L^\infty_x(\mathbb{R}^d)}.$$

For all sufficiently large $n$, the following estimates hold for all $t \in [0,T]$:

$$\|\nabla v_n(t)\|_{L^\infty} \leq V_n(t),$$

$$\|Y_n(t)\|_{L^\infty} \leq \|Y_0\|_{L^\infty} e^{\int_0^t V_n(s) \, ds},$$

$$\|Y_n \cdot \nabla v_n(t)\|_{C^\alpha} \leq C(T) \left( V_n(t) \|Y_n(t)\|_{C^\alpha} + 1 \right),$$

$$\|Y_n(t)\|_{C^\alpha} \leq C(T) \|Y_0\|_{C^\alpha} e^{\int_0^t C(T) V_n(s) \, ds},$$

(9.1) (9.2) (9.3) (9.4)
where \( C(T) \) depends continuously upon \( F, \| \rho_0 \|_{L^\infty} \) and \( T \). Moreover,
\[
V_n(t) \leq C(T)(1 - \log r) + C(T) e^{C(T) \int_0^t V_n(s) \, ds} r^\alpha \text{ for any } r \in (0, 1].
\]

\[ (9.5) \]

Proof. Because \( \| \rho_{n,0} \|_{L^\infty} \leq \| \rho_0 \|_{L^\infty}, \) we see that all solutions will exist on the time interval \([0, T]\). It also follows that \((v_n)\) is uniformly bounded in the log-Lipschitz norm.

To streamline notation, we drop the subscript \( n \) throughout the rest of this subsection.

In what follows, \( Y \) will always be a vector field among the first \( d - 1 \) vector fields of a frame in \( \mathcal{V} \). Except in the proof of \((9.5)\), this frame will be arbitrary. Taking the supremum of the bounds over all the frames gives the final estimates expressed in terms of \( \mathcal{V} \).

Equation \((9.1)\) follows immediately from Proposition 6.1. Equation \((9.2)\) follows from \((8.2)\) and Grönwall’s Lemma. By Corollary 6.3,
\[
\| Y \cdot \nabla v(t) \|_{C^\alpha} \leq CV(t) \| Y \|_{C^\alpha} + \| \text{div}(\rho Y) \|_{L^\infty}
\]
\[
\leq CV(t) \| Y \|_{C^\alpha} + \| Y \cdot \nabla \rho \|_{L^\infty} + \rho \| \text{div} Y \|_{L^\infty} \leq CV(t) \| Y \|_{C^\alpha} + C(T),
\]
where we used \((8.7)\). This gives \((9.3)\).

The derivations of \((9.4)\) and \((9.5)\) are more involved. We argue much as in Sections 10.2 and 10.3 of [1], which follows an argument in [20], but the differences with [1] are too subtle to merely outline, so we include their complete derivations.

\[(9.4): \text{Bound on } Y \text{ in } C^\alpha. \] We write \((8.2)\) as
\[
\frac{d}{dt} Y(t, \eta(t, x)) = (Y \cdot \nabla v)(t, \eta(t, x)),
\]
and integrate in time, to obtain
\[
Y(t, \eta(t, x)) = Y_0(x) + \int_0^t (Y \cdot \nabla v)(s, \eta(s, x)) \, ds.
\]
This leads to the simple bound,
\[
\| Y(t) \|_{L^\infty} \leq \| Y_0 \|_{L^\infty} + \int_0^t \| (Y \cdot \nabla v)(s) \|_{L^\infty} \, ds.
\]
\[ (9.6) \]

Using the inverse flow map, we also have
\[
Y(t, x) = Y_0(\eta^{-1}(t, x)) + \int_0^t (Y \cdot \nabla v)(s, \eta(s, \eta^{-1}(t, x))) \, ds.
\]
\[ (9.7) \]

First, let us estimate \( \| \nabla(\eta(s, \eta^{-1}(t, x))) \|_{L^\infty} \). We start with
\[
\partial_s \eta(s, \eta^{-1}(t, x)) = v(s, \eta(s, \eta^{-1}(t, x))).
\]
Then
\[
\partial_s \nabla (\eta(s, \eta^{-1}(t, x))) = \nabla v(s, \eta(s, \eta^{-1}(t, x))) \nabla(\eta(s, \eta^{-1}(t, x))).
\]
Integrating in time and using \( \nabla(\eta(s, \eta^{-1}(t, x))) \big|_{s=t} = I, \) the identity matrix, we have
\[
\nabla (\eta(s, \eta^{-1}(t, x))) = I - \int_s^t \nabla v(\tau, \eta(\tau, \eta^{-1}(t, x))) \nabla(\eta(\tau, \eta^{-1}(t, x))) \, d\tau.
\]
By Grönwall’s lemma, then,
\[
\| \nabla(\eta(s, \eta^{-1}(t, x))) \|_{L^\infty} \leq \exp \int_s^t \| \nabla v(\tau) \|_{L^\infty} \, d\tau \leq \exp \int_s^t V(\tau) \, d\tau.
\]
Thus, applying (3.2) and (9.3) to (9.7), and using (3.2) and (8.9), and adding the bound in (9.6), we see that

\[ \|Y(t)\|_{C^\alpha} \leq \|Y_0\|_{C^\alpha} \|\nabla \eta^{-1}(t)\|_{L^\infty}^\alpha + \int_0^t \|Y \cdot \nabla v\|_{C^\alpha} \|\nabla \eta(s, \eta^{-1}(t, x))\|_{L^\infty}^\alpha \, ds \]

\[ \leq \|Y_0\|_{C^\alpha} \exp\left(\alpha \int_0^t V(\tau) \, d\tau\right) \]

\[ + C(T) \int_0^t (V(s) \|Y(s)\|_{C^\alpha} + 1) \exp\left(\alpha \int_s^t V(\tau) \, d\tau\right) \, ds. \]

Letting

\[ y(t) = \|Y(t)\|_{C^\alpha} \exp\left(-\int_0^t V(\tau) \, d\tau\right), \]

we see that

\[ y(t) \leq \|Y_0\|_{C^\alpha} + C(T) \exp\left(-\int_0^t V(\tau) \, d\tau\right) \int_0^t \left(V(s) y(s) \exp\left(\int_0^s V(\tau) \, d\tau + 1\right) \right. \]

\[ \exp\left(\alpha \int_s^t V(\tau) \, d\tau\right) \, ds \]

\[ \leq \|Y_0\|_{C^\alpha} + C(T) \int_0^t V(s) y(s) \, ds \]

\[ + C(T) \exp\left(-\int_0^t V(\tau) \, d\tau}\right) \int_0^t \exp\left(\alpha \int_s^t V(\tau) \, d\tau\right) \, ds \]

\[ \leq \|Y_0\|_{C^\alpha} + C(T) \int_0^t V(s) y(s) \, ds + C(T) \leq C(T) \int_0^t V(s) y(s) \, ds. \]

It follows from Grönwall’s lemma that \( y(t) \leq \|Y_0\|_{C^\alpha} \exp\left(C(T) \int_0^t V(s) \, ds\right) \) and hence that

\[ \|Y(t)\|_{C^\alpha} \leq \|Y_0\|_{C^\alpha} \exp\left(C(T) \int_0^t V(s) \, ds\right), \]

which is (9.4).

\textbf{(9.5): Bound on} \( V \). Fix \( x \in \mathbb{R}^d \). We start by splitting the principal value integral part of \( \nabla v \) (as in Proposition 6.1) into two parts as

\[ - \text{p.v.} \int \nabla \nabla \Phi(x - y) \rho(y) \, dy = - \text{p.v.} \int \nabla(a_r \nabla \Phi)(x - y) \rho(y) \, dy \]

\[ - \text{p.v.} \int \nabla((1 - a_r) \nabla \Phi)(x - y) \rho(y) \, dy, \]

where \( r \in (0, 1) \) is arbitrary.

On the support of \( \nabla(1 - a_r) = -\nabla a_r, |x - y| \leq 2r \), so

\[ |\nabla((1 - a_r) \nabla \Phi)| \leq |(1 - a_r) \nabla \nabla \Phi| + |(-\nabla a_r) \otimes \nabla \Phi| \leq C |x - y|^{-d}. \]
Hence, one term in (9.8) is easily bounded, using only that $\rho(t) \in L^1 \cap L^\infty$, by
\[
\left| \text{p.v.} \int \nabla((1 - a_r)\nabla \Phi)(x - y)\rho(t, y) \, dy \right| \leq C \int_{B^c_\rho(x)} |x - y|^{-d} |\rho(t, y)| \, dy
\]
\[
\leq C \int_{r}^{1} \frac{\|\rho(t)\|_{L^\infty}}{\lambda^d} \lambda^{d-1} \, d\lambda + C \|x - \cdot\|_{L^\infty(B^c_\rho(x))} \|\rho_0\|_{L^1}
\leq -C(T) \log r \|\rho_0\|_{L^\infty} + C(T) \|\rho_0\|_{L^1} \leq C(T) (-\log r + 1).
\] (9.10)

We used (1.10) to obtain the constant, $C(T)$.

For the other term in (9.8), letting $\mu_{rh}$ be as in Definition 3.2, we can write
\[
\left| \text{p.v.} \int \nabla(a_r \nabla \Phi)(x - y)\rho(t, y) \, dy \right| = \left| \lim_{h \to 0} \nabla(\mu_{hr} \nabla \Phi) \ast \rho(t, x) \right| = \lim_{h \to 0} |B|,
\]
where
\[
B(t, x) := \nabla(\mu_{rh} \nabla \Phi) \ast \rho \big|_{(t, x)}.
\]

Because $\mathcal{Y}_0$ is a sufficient family, there exists a frame $Y_0$ in $\mathcal{Y}_0$ so that $|Y_{0,d}(x)| \geq I(\mathcal{Y}_0)/2$. Applying Lemma 5.1 to this frame gives, at $(t, x)$,
\[
|B| \leq \frac{P(Y_1, \ldots, Y_{d-1})}{|Y^d_0|^4} \sum_{i=j}^{d-1} |BY_j| + \text{tr } B,
\]
where $P$ has degree $n_d$. Moreover, from Proposition 8.5, it follows, using (reverse) Grönwall’s Lemma, that
\[
|Y_{d}(t, \eta(t, x))| \geq |Y_{0,d}(x)| e^{-\int_{0}^{t} (V(s) + \|\rho(s)\|_{L^\infty}) \, ds} \geq I(\mathcal{Y}_0) e^{-t\|\rho_0\|_{L^\infty}} e^{-\int_{0}^{t} V(s) \, ds}. \tag{9.11}
\]

In light of the bound in (9.2), we have (recall from Lemma 5.1 that $n_d = 4d - 5$)
\[
|B| \leq C \frac{\|Y(t)\|_{L^\infty}}{I(\mathcal{Y}_0)^4} e^{2\int_{0}^{t} (V(s) + \|\rho(s)\|_{L^\infty}) \, ds} \sum_{i=j}^{d-1} |BY_j| + |\text{tr } B|
\leq C(T) e^{(n_d + 4) \int_{0}^{t} V(s) \, ds} \sum_{i=j}^{d-1} |BY_j| + |\text{tr } B|. \tag{9.12}
\]

We first compute $\text{tr } B$. We have,
\[
\text{tr } B = \sum_{j=1}^{d} [\partial_j \mu_{rh} \partial_j \Phi] \ast \rho + [\mu_{rh} \Delta \Phi] \ast \rho = \sum_{j=1}^{d} [\partial_j \mu_{rh} \partial_j \Phi] \ast \rho,
\]
using $\Delta \Phi = \delta_0$ and $\mu_{rh}(0) = 0$ to remove the one term. By (3.3), we have for $j = 1, 2, \ldots, d$,
\[
|[\partial_j \mu_{rh} \partial_j \Phi] \ast \rho| \leq C \int_{r}^{2r} \frac{|\rho(t, y)|}{|x - y|^{d-1}} \, dy + C \int_{h}^{2h} \frac{|\rho(t, y)|}{|x - y|^{d-1}} \, dy
\leq C \frac{r}{r} \frac{\|\rho(t)\|_{L^\infty} \lambda^{d-1} \, d\lambda}{\lambda^{d-1}} + C \frac{2h}{h} \frac{\|\rho(t)\|_{L^\infty} \lambda^{d-1} \, d\lambda}{\lambda^{d-1}}
= C \frac{\|\rho(t)\|_{L^\infty} \lambda^{d-1} \, d\lambda}{\lambda^{d-1}} \leq C(T),
\]
where we used (1.10). Thus,
\[
\lim_{h \to 0} |\text{tr } B| \leq C(T).
\]
We next estimate \(|BY_j|\). Because
\[ B^k_l = \partial_l [\mu_r \partial_k \Phi] * \rho, \]
we have, for \( j < d \),
\[ (BY_j)^k = B^k_l Y_j^l = (\partial_l [\mu_r \partial_k \Phi] * \rho)Y_j^l \]
\[ = (\partial_l [\mu_r \partial_k \Phi] * \rho)Y_j^l - \partial_l [\mu_r \partial_k \Phi] * (\rho Y_j^l) + (\mu_r \partial_k \Phi) * \text{div}(\rho Y_j) \]
\[ =: \sum_{l=1}^d d^k_l + (\mu_r \partial_k \Phi) * \text{div}(\rho Y_j). \]

By Lemma 3.3,
\[ |d^k_l| = \left| \int \nabla [\mu_r \nabla \Phi](x - y)(Y_j(x) - Y_j(y))\rho(y) \, dy \right| \leq C \|Y_j(t)\|_{C^\alpha} \|\rho_0\|_{L^\infty} \, r^\alpha, \]
and using (1.10) and (9.4), we see that
\[ \sum_{l=1}^d \lim_{h \to 0} |d^k_l| \leq C \|Y_j(t)\|_{C^\alpha} \|\rho_0\|_{L^\infty} \, r^\alpha \leq C e^{C(T) \int_0^t V(s) \, ds} \, r^\alpha. \]

Also by Lemma 3.3 along with (8.7),
\[ \lim_{h \to 0} (\mu_r \partial_k \Phi) * \text{div}(\rho Y_j) = \lim_{h \to 0} \left| \int_{\mathbb{R}^d} (\mu_r \nabla \Phi)(x - y) \text{div}(\rho Y_j)(y) \, dy \right| \]
\[ \leq C \|\text{div}(\rho Y_j)\|_{L^\infty} \, r^\alpha \leq C \|\nabla_0 \cdot \nabla \rho_0\|_{L^\infty} e^{C(T)T} \, r^\alpha \leq C(T) \, r^\alpha. \]

Thus, returning to (9.12), and noting that our bounds are independent of \( h \), we have (now including the explicit dependence of \( B \) on \( t \) and \( x \)),
\[ \lim_{h \to 0} |B(t, x)| \leq C(T) \, r^\alpha e^{C(T) \int_0^t V(s) \, ds} + C(T). \]

Combined with (9.10), which applies for all \( x \in \mathbb{R}^d \), we obtain,
\[ V(t) \leq C(1 - \log r) \|\rho_0\|_{L^1 \cap L^\infty} + \sup_{Y_0} \sup_{x \in \mathbb{R}^d} \lim_{h \to 0} |B(t, x)| \]
\[ \leq C(T)(1 - \log r) + C(T)e^{C(T) \int_0^t V(s) \, ds} \, r^\alpha, \]
which is (9.5).

9.2. Proof of Theorem 2.3. To prove Theorem 2.3, we will first close the bound on \( V_n(t) \) as in Section 10.4 of [1] to obtain a uniform bound on it, and so on \( \|\nabla v_n\|_{L^\infty} \). We will then show that our sequence of approximate solutions converges to a solution to the aggregation equation having the same uniform bounds.

Closing the bounds. Setting
\[ r = r_n = \exp \left( -\frac{C(T)}{\alpha} \int_0^t V_n(s) \, ds \right), \]
we have
\[ 1 - \log r = 1 + \frac{C(T)}{\alpha} \int_0^t V_n(s) \, ds, \quad r^\alpha = \exp \left( -C(T) \alpha \int_0^t V_n(s) \, ds \right). \]
The bound in (9.5) thus yields the estimate,

\[ V_n(t) \leq C(T) + \frac{C(T)}{\alpha} \int_0^t V_n(s) \, ds. \] (9.13)

By Grönwall’s lemma, we conclude that

\[ \|\nabla v_n(t)\|_{L^\infty} \leq V_n(t) \leq C(T)e^{C(T)\alpha^{-1}t} \]

so also, by virtue of (9.4),

\[ \|\gamma_n(t)\|_{C^\alpha} \leq C(T)\exp\left(C(T)e^{C(T)\alpha^{-1}t}\right). \] (9.14)

**Convergence to a solution.** What is left to show is that the sequence of regular solutions \((\rho_n)\) converges to a weak solution \(\rho\) and that key bounds in Proposition 9.1 continue to hold for the limiting solution. In proving this, we adapt the corresponding arguments for the 2D Euler equations, combining aspects of the arguments in Section 8.2 of [17], pages 105-106 of [9], and Section 10.5 of [1]. This consists of the following steps:

(i) Prove that the sequence of forward flow maps \((\eta_n)\) converges to a function \(\eta\) and that the sequence of inverse flow maps \((\eta_n^{-1})\) converges to \(\eta^{-1}\) in \(L^\infty([0, T] \times \mathbb{R}^d)\).

(ii) Define \(\rho(t, x)\) to satisfy (1.8) and show that \(\rho_n \to \rho\) in \(L^\infty([0, T]; L^p(\mathbb{R}^d))\) for any \(p \in [1, \infty)\).

(iii) Define \(v = -\nabla \Phi * \rho\) and show that \(\eta\) is, in fact, the flow map for \(v\). Hence, \(\rho\) is a Lagrangian solution to (1.1, 1.4).

(iv) Show that \(\nabla v \in L^\infty([0, T] \times \mathbb{R}^d)\), giving (a) in Theorem 2.3.

(v) Show that \(\gamma \in L^\infty(0, T; C^\alpha(\mathbb{R}^d))\).

(vi) Show that \(\rho\) is also a weak Eulerian solution.

(vii) Show (b) and (c) in Theorem 2.3.

(i). By (8.9),

\[ \|\nabla \eta_n(t)\|_{L^\infty}, \|\nabla \eta_n^{-1}(t)\|_{L^\infty} \leq J(t) \leq e^{V_n(t)} \leq C(T). \]

Moreover, for all \(t_1, t_2 \in [0, T]\),

\[ |\eta_n(t_1, x) - \eta_n(t_2, x)| \leq \|v_n\|_{L^\infty([0, T] \times \mathbb{R}^d)} |t_1 - t_2| \leq C(T) |t_1 - t_2|. \]

That is, the sequence \((\eta_n)\) is Lipschitz-continuous in time and space uniformly in \(n\), so the Arzelà-Ascoli theorem shows that a subsequence converges uniformly on compact subsets to a Lipschitz-continuous in time and space function, \(\eta\). The analogous argument gives the same type of convergence for the inverse flow maps \(\eta_n^{-1}\) to \(\eta^{-1}\) (though with lower time regularity, an issue that will not affect us). With the compact support of the initial data it easy to see that a subsequence converges in \(L^\infty([0, T] \times \mathbb{R}^d)\). Once we prove that in the limit we obtain an Eulerian solution to (1.1, 1.4), for which we have uniqueness, it will follow that these convergences are for the full sequence; for now, we will simply reindex as needed to avoid additional notation.

(ii). For any fixed \(x \in \mathbb{R}^d\), let \(\overline{\rho}(t, x)\) be the solution to the single-dimensional ODE, \(\partial_t \overline{\rho}(t, x) = F(\overline{\rho}(t, x)), \overline{\rho}(0, x) = \rho_0(x)\) on \([0, T]\), and define \(\rho\) on \([0, T] \times \mathbb{R}^d\) by \(\rho(t, x) = \overline{\rho}(t, x)\).
\( \Phi(t, \eta^{-1}(t, x)) \). It follows that (1.8) holds. Therefore,
\[
\| \rho(t, \eta(t, \cdot)) - \rho_n(t, \eta_n(t, \cdot)) \| \leq \| \rho_0 - \rho_{0n} \| + \int_0^t \| F(\rho(s, \eta(s, \cdot))) - F(\rho_n(s, \eta_n(s, \cdot))) \| \, ds
\]
\[
\leq \| \rho_0 - \rho_{0n} \| + C(T) \int_0^t \| \rho(s, \eta(s, \cdot)) - \rho_n(s, \eta_n(s, \cdot)) \| \, ds,
\]
where we note that \( \| \cdot \| \) is the \( L^2 \) norm and we used that \( \rho(\eta(s, \cdot)) \) and \( \rho_n(s, \eta_n(s, \cdot)) \) are supported in some fixed compact set for all \( s \in [0, T] \) because of the compact support of the initial data and the uniform bound on \( \rho_n \) in \( L^\infty([0, T] \times \mathbb{R}^d) \). But \( \| \rho_0 - \rho_{0n} \| \to 0 \) as \( n \to \infty \) by (ii) of Proposition 7.1 with \( p = 2 \), so Grönwall’s lemma gives that
\[
\| \rho(t, \eta(t, \cdot)) - \rho_n(t, \eta_n(t, \cdot)) \| \leq \| \rho_0 - \rho_{0n} \| e^{C(T)t} \to 0. \tag{9.15}
\]
A similar argument applied to the inverse flow map gives
\[
\| \rho(t) - \rho_n(t) \| \leq \| \rho_0(\eta^{-1}(t, \cdot)) - \rho_{0n}(\eta_n^{-1}(t, \cdot)) \|
\]
\[
+ \int_0^t \| F(\rho(s, \eta(s, \eta^{-1}(s, \cdot))) - F(\rho_n(s, \eta_n(s, \eta_n^{-1}(s, \cdot))) \| \, ds. \tag{9.16}
\]
Now,
\[
\| \rho_0(\eta^{-1}(t, \cdot)) - \rho_{0n}(\eta_n^{-1}(t, \cdot)) \|
\]
\[
\leq \| \rho_0(\eta^{-1}(t, \cdot)) - \rho_0(\eta_n^{-1}(t, \cdot)) \| + \| \rho_0(\eta_n^{-1}(t, \cdot)) - \rho_{0n}(\eta_n^{-1}(t, \cdot)) \|. 
\]
The first term vanishes in the limit by Lemma 9.3, below, the second vanishes because \( \rho_{0n} \to \rho_0 \) in \( L^2 \) and we have a uniform bound on the Jacobian, \( \det \nabla \eta_n^{-1} \), by Proposition 8.7.

For the time integral in (9.16), we have
\[
\| F(\rho(s, \eta(s, \eta^{-1}(s, \cdot))) - F(\rho_n(s, \eta_n(s, \eta_n^{-1}(s, \cdot))) \| \]
\[
\leq C(T) \| \rho(s, \eta(s, \eta^{-1}(s, \cdot))) - \rho_n(s, \eta_n(s, \eta_n^{-1}(s, \cdot))) \|
\]
\[
\leq C(T) \| \rho(s, \eta(s, \eta_n^{-1}(s, \cdot))) - \rho_n(s, \eta_n(s, \eta_n^{-1}(s, \cdot))) \|
\]
\[
+ C(T) \| \rho(s, \eta(s, \eta_n^{-1}(s, \cdot))) - \rho(s, \eta(s, \eta_n^{-1}(s, \cdot))) \|. 
\]
Again using that the Jacobian for the inverse map \( \eta_n^{-1} \) is bounded, we have
\[
\| \rho(s, \eta(s, \eta_n^{-1}(s, \cdot))) - \rho_n(s, \eta_n(s, \eta_n^{-1}(s, \cdot))) \| \leq C(T) \| \rho(s, \eta(s, \cdot)) - \rho_n(s, \eta_n(s, \cdot)) \|,
\]
which vanishes in the limit by (9.15). Letting \( f(s, x) = \rho(s, \eta(s, x)) \), we can write,
\[
\| \rho(s, \eta(s, \eta^{-1}(s, \cdot))) - \rho(s, \eta(s, \eta_n^{-1}(s, \cdot))) \| = \| f(\eta^{-1}(s, \cdot)) - f(\eta_n^{-1}(s, \cdot)) \|,
\]
which vanishes in the limit by Lemma 9.3. We conclude from the dominated convergence theorem that the time integral in (9.16) vanishes in the limit. Noting that the arguments above using the \( L^2 \) norm apply for any \( p \in [1, \infty) \), we see that
\[
\rho_n(t) \to \rho(t) \text{ in } L^\infty(0, T; L^p(\mathbb{R}^d)) \text{ for any } p \in [1, \infty). \tag{9.17}
\]

(iii). Define \( v = -\nabla \Phi \ast \rho \). Then since \( \rho_n(t) \) and \( \rho(t) \) are compactly supported in some common \( K(T) \subset \mathbb{R}^d \) for all \( t \in [0, T] \), we have, for any \( p \in (2, \infty) \) and \( p' = p/(p-1) \),
\[
\| v_n - v \|_{L^\infty([0,T];L^\infty(\mathbb{R}^d))} \leq \| \nabla \Phi \|_{L^{p'}(2K(T))} \| \rho_n - \rho \|_{L^\infty(0,T;L^p(\mathbb{R}^d))} \to 0. \tag{9.18}
\]
We see in the same way that \((v_n)\) is uniformly bounded in \(L^\infty([0,T] \times \mathbb{R}^d)\); hence, we can apply the dominated convergence theorem to give, for any \((t,x) \in [0,T] \times \mathbb{R}^d\),

\[
\eta(t,x) - x = \lim_{n \to \infty} \eta_n(t,x) = \lim_{n \to \infty} \int_0^t v_n(s, \eta_n(s,x)) \, ds = \int_0^t \lim_{n \to \infty} v_n(s, \eta_n(s,x)) \, ds.
\]

But,

\[
|v_n(s, \eta_n(s,x)) - v(s, \eta(s,x))| \\
\leq |v_n(s, \eta_n(s,x)) - v(s, \eta_n(s,x))| + |v(s, \eta_n(s,x)) - v(s, \eta(s,x))| \leq \|v_n(s) - v(s)\|_{L^\infty} + |v(s, \eta_n(s,x)) - v(s, \eta(s,x))|.
\]

The first term on the right vanishes by (9.18); the second vanishes because \(v\) has a log-Lipschitz modulus of continuity and \(\eta_n \to \eta\) in \(L^\infty([0,T] \times \mathbb{R}^d)\). In the limit, then, we see that

\[
\eta(t,x) - x = \int_0^t v(s, \eta(s,x)) \, ds.
\]

We conclude that \(\eta\) is the flow map for \(v\) and hence \(\rho\) is a Lagrangian solution to (1.1, 1.4).

**iv.** Using the Arzelà-Ascoli theorem as in (i), we know that for all \(t \in [0,T]\) the sequence \((v_n)\) converges uniformly on compact subsets to a Lipschitz-continuous vector field. It follows from (iii) and the uniqueness of limits that that limit is \(v(t)\); hence, \(\nabla v \in L^\infty([0,T] \times \mathbb{R}^d)\).

**v.** Let \(Y_0 = Y_0^{(j)}\), \(k < d\), be any vector field from among the first \(d-1\) vector fields in a frame of \(\gamma_0\). We can write (8.1) in the form, \(Y_0 \cdot \nabla \eta_n = Y_n \circ \eta_n\). Then by (8.9) and (9.4) with (3.2),

\[
\|Y_0 \cdot \nabla \eta_n\|_{C^\alpha([0,T] \times \mathbb{R}^d)} \leq \|Y_0\|_{C^\alpha([0,T] \times \mathbb{R}^d)} \|\nabla \eta_n\|_{L^\infty([0,T] \times \mathbb{R}^d)} \leq C(T) \|Y_0\|_{C^\alpha} \exp \left( C(T) \int_0^t V_n(s) \, ds \right),
\]

so \(Y_0 \cdot \nabla \eta_n\) is uniformly bounded in \(C^\alpha(\mathbb{R}^d)\). But for any compact subset \(K\) of \(\mathbb{R}^d\), \(C^\alpha(K)\) is compactly embedded in \(C^\beta(K)\) for all \(\beta < \alpha\) so a subsequence of \((Y_0 \cdot \nabla \eta_n(t))\) converges in \(C^\beta(K)\) to some \(f_K(t)\) for all \(\beta < \alpha\). Letting \(K_n\) be the closed ball of radius \(n\) centered at the origin and making a diagonalization argument, we can obtain a function \(f(t) \in C^\alpha(\mathbb{R}^d)\) for which a subsequence of \((Y_0 \cdot \nabla \eta_n(t))\) converges in \(C^\alpha_{loc}(\mathbb{R}^d)\) to \(f(t)\) for all \(\beta < \alpha\).

To show that \(f(t) = Y_0 \cdot \nabla \eta(t)\), we need only show convergence of \(Y_0 \cdot \nabla \eta_n(t) \to Y_0 \cdot \nabla \eta(t)\) in some weaker sense. Let \(\varphi \in C^\infty_c(\mathbb{R}^d)\). As in (2.4) (noting that \(Y(t), \text{div} Y(t) \in C^\alpha\)),

\[
(Y_0 \cdot \nabla \eta_n(t), \varphi) = -(\eta_n(t), \text{div}(\varphi Y_0)) = -(\eta(t), \text{div}(\varphi Y_0)) = (Y_0 \cdot \nabla \eta(t), \varphi),
\]

by the strong convergence of \(\eta_n\) to \(\eta\) we established earlier. Hence, \(Y_0 \cdot \nabla \eta_n(t) \to Y_0 \cdot \nabla \eta(t)\) as distributions, so we can conclude that \(f(t) = Y_0 \cdot \nabla \eta(t)\). Since \(f(t) \in C^\alpha(\mathbb{R}^d)\), we see that

\[
Y(t) = (Y_0 \cdot \nabla \eta) \circ \eta^{-1} = f(t) \circ \eta^{-1}
\]

lies in \(C^\alpha(\mathbb{R}^d)\) by (3.2).

**vi.** That \(\rho\) is a weak Eulerian solution to (1.1, 1.4) as in Definition 1.3 follows easily from the convergences we proved of the sequences \((\rho_n)\) and \((v_n)\).
(vii). Now let \( \psi_j(t) \) be \( \psi_j \) transported by the flow map. Because \( \psi_j(t, x) = \psi_j(0)(\eta^{-1}(t, x)) \), we have \( \|\psi_j(t)\|_{L^\infty} = \|\psi_j(0)\|_{L^\infty} \). Taking the gradient of the transport equation, \( \partial_t \psi_j + v \cdot \nabla \psi_j = 0 \), we have

\[
\partial_t \nabla \psi_j + \nabla (v \cdot \nabla \psi_j) = 0.
\]

(Note that we have assumed insufficient regularity on \( \psi_j \) for the second term here. We could proceed either by using the sequence of approximating flow maps as we did above, or by making a separate approximation argument. For simplicity, however, we present this somewhat formal approach.) But

\[
(\nabla (v \cdot \nabla \psi_j))^i = \partial_i (v^k \partial_k \psi_j) = v^k \partial_k \partial_i \psi_j + \partial_i v^k \partial_k \psi_j = v \cdot \nabla (\nabla \psi_j)^i + ((\nabla v)^T \nabla \psi_j)^i
\]

since \( \nabla v \) is symmetric. Hence,

\[
\partial_t \nabla \psi_j + v \cdot \nabla \nabla \psi_j = -\nabla v \cdot \nabla \psi_j,
\]

which we can rewrite as

\[
\nabla \psi_j(t, x) = \nabla \psi_j(0, \eta^{-1}(t, x)) - \int_0^t (\nabla v \nabla \psi_j)(s, \eta(s, \eta^{-1}(t, x))) \, ds.
\]

Therefore,

\[
\|\nabla \psi_j(t)\|_{L^\infty} \leq \|\nabla \psi_j(0)\|_{L^\infty} + \int_0^t \|\nabla v\|_{L^\infty} \|\nabla \psi_j(s)\|_{L^\infty} \, ds.
\]

Applying Grönnwall’s lemma gives (2.1).

Noting that the level sets of both \( \rho_0 \) and \( \psi_j \) are transported by the flow map, we see that \( \psi_j(t) \) remains level-set-compatible to \( \rho(t) \) on the interior of \( \text{supp} \psi_j(t) \). Finally, note that \( Z_j \in L^\infty(0, T; C^\alpha(\mathbb{R}^d)) \) and is non-vanishing, which follow from Proposition 8.5 since \( Z_j = Y_{d,j} \) and each frame lies in \( C^\alpha(\mathbb{R}^d) \). Moreover, by Proposition 8.6, \( Z_j \) is parallel to \( \nabla \psi_j(t) \) for all \( t \in [0, T] \). Therefore, we see that the level sets of \( \psi_j(t) \) retain their \( C^{1,\alpha} \) regularity.

**Remark 9.2.** It follows, much as in the proof of (v) above, that \( \mathcal{Y} \cdot \nabla v \in C^\alpha(\mathbb{R}^d) \).

In the proof above of Theorem 2.3, we used Lemma 9.3, a version of continuity of the \( L^p \) norm with respect to translation.

**Lemma 9.3.** Let \( f \in L^p(\mathbb{R}^d), \ p \in [1, \infty), \) be compactly supported, let \( (g_n) \) be a sequence of bounded homeomorphisms on \( \mathbb{R}^d \), and let \( g \) be a bounded homeomorphism on \( \mathbb{R}^d \). Assume that \( g_n \to g \) uniformly on compact subsets of \( \mathbb{R}^d \). Then

\[
f \circ g_n \to f \circ g \text{ in } L^p(\mathbb{R}^d) \text{ as } n \to \infty.
\]

**Proof.** This is a simple adaptation of the classical proof in which \( g_n \) is translation and \( g \) is the identity. See, for instance, Theorem 8.19 p. 134-135 of [21]. Or see Lemma 8.2 of [15] for an explicit proof.

9.3. **Proof of Corollary 2.4.** Apply Corollary 7.2 to Theorem 2.3. Since each \( Z_j(t) \) is parallel to \( \nabla \psi_j(t) \), each is normal to the boundary; because \( Z_j(t) \in C^\alpha \) it follows that \( \partial \Omega(t) \) is \( C^{1,\alpha} \).
10. Removability of singularities: proof of Theorem 2.5

We establish in this section the higher regularity of the corrected velocity gradient, constructing the matrix \( A \) of Theorem 2.5.

**Proof of Theorem 2.5.** Let \( Y_0 \) be the sufficient family given by Proposition 7.1 with \( Y \) its pushforward constructed in the proof of Theorem 2.3. First, we argue locally. Let \( Y = (Y_1, \ldots, Y_d) \) be any frame in \( Y \) for which \(|Y_d| > I(Y)/2\) on some nonempty open \( \Omega \subseteq \mathbb{R}^d \). Define

\[
A := -\frac{1}{|Y_d|^2} Y_d \otimes Y_d = -\hat{Y}_d \otimes \hat{Y}_d \text{ on } \Omega, \tag{10.1}
\]

where \( \hat{Y}_d = Y_d/|Y_d| \), recalling that \( Y_d = \wedge_{k<d} Y_k \).

For all \( i, k \),

\[
[(Y_d \otimes Y_d)Y_k]^i = Y_d^i Y_d^j Y_k^j = Y_d^i (Y_d \cdot Y_k) = \delta_{dk} Y_d^i |Y_d|^2,
\]

so

\[
AY_k = 0 \text{ for } k < d, \quad AY_d = -Y_d.
\]

For \( k < d \), then,

\[
(\nabla v - \rho A)Y_k = \nabla v Y_k - \rho AY_k = \nabla v Y_k \in C^\alpha,
\]

since \( \nabla v Y_k \in C^\alpha \) as noted in Remark 9.2. Because \( \text{tr } A = -1 \),

\[
\text{tr}(\nabla v - \rho A) = \text{div } v - \rho \text{ tr } A = -\rho(1 + \text{tr } A) = 0.
\]

Since \( \nabla v - \rho A \) is symmetric and traceless, it follows from Lemma 5.2 that \( \nabla v - \rho A \in C^\alpha(\Omega) \).

The result then follows by using a partition of unity and the compact support of \( \rho_0 \) to combine each local version of \( A \).

**Remark 10.1.** Because \( Y_d^{(j)} \) is the wedge product of \( d - 1 \) vector fields lying in \( C^\alpha \), it also lies in \( C^\alpha \). Nonetheless, if \( u \) is any nonvanishing vector field parallel to \( Y_d \) on \( U_j \), even a highly irregular one, then we see from (10.1) that \( A_j = -\hat{u} \otimes \hat{u} \), where \( \hat{u} = u/|u| \). In particular, transporting \( \psi_j \) to give \( \psi_j(t) \) as in Theorem 2.3, we can use \( u = \nabla \psi_j \) locally, so \( A = -\nabla \psi_j \otimes \nabla \psi_j \), even though \( \nabla \psi_j(t) \) is only in \( L^\infty \).

**Proof of Theorem 2.6.** The result follows from (8.8)\(_1\) for \( k = 0 \) and the bound on \( J(t) \) that follows from the proof of Theorem 2.3 (or follows easily from (a) of its statement).

Now assume \( k = 1 \). By Theorem 2.5, \( \nabla v - \rho A \in C^\alpha(\eta(t, W)) \). But \( A \in C^\alpha(\mathbb{R}^d) \) and \( \rho \in C^\alpha(\eta(t, W)) \) by the \( k = 0 \) result. We conclude that \( \nabla v \in C^\alpha(\eta(t, W)) \) for all \( t \in [0, T] \). The result then follows from (8.8)\(_3\) and Lemma 8.4.

11. Patch boundary regularity: comparison to \([3]\)

We now give a brief explanation of how Corollary 2.4 is established in \([3]\) then compare it to our approach.

The authors of \([3]\) first make the transformation,

\[
s := -\frac{1}{T} \log \left( 1 - \frac{t}{T} \right), \quad \bar{\rho}(s, x) := T(T - t) \rho(t, x), \quad \bar{v}(s, x) := T(T - t) v(t, x).
\]

Relabeling to return to \( t, \rho, v \), we obtain (for \( F(\rho) = \rho^2 \)), in place of (1.1, 1.4),

\[
\partial_t \rho + v \cdot \nabla \rho = \rho(\rho - T), \quad v = -\nabla \Phi \ast \rho, \quad \rho(0) = \rho_0.
\]
Hence, $F(\rho) = \rho^2$ is transformed to $F(\rho) = \rho(\rho - T)$, the negative of the logistic map, without changing anything else. It is easy to see that if $0 \leq \rho_0 \leq T$ then $0 \leq \rho(t) \leq T$ for all $t \in [0, T)$. Moreover, when $\rho_0 = 1_\Omega$, since $T^* = 1$, we can set $T = 1$, and it is easy to see that the forcing vanishes for all $t \in [0, T)$. Hence, for the special case of patch data, the density is purely transported in the transformed variables. All further arguments in [3] apply then to $\dot{\rho} + v \cdot \nabla \rho = 0$. Or, in our notation, the authors treat (1.1, 1.4) in the special case, $F \equiv 0$.

Now, even though $\psi_j(0) \in C^{1,\alpha}(\mathbb{R}^d)$, it is clear from (9.20) that we cannot propagate more than the Lipschitz regularity of $\psi_j$. The authors of [3] get around this difficulty by evolving each $\psi_j(0)$ to obtain the function $\phi(t)$ with the nonsingular forcing term $-\phi 1_\Omega = -\phi$:

$$\partial_t \phi + v \cdot \nabla \phi = -\phi \rho, \quad \phi(0) = \phi_0 := \psi_j(0).$$

Here, $\phi_0$ defines one component of $\partial \Omega$, where it vanishes and has non-vanishing gradient.

Setting $Z = \nabla \phi$, and using that $\phi \nabla \rho \equiv 0$, a calculation like that in (9.20) gives

$$\partial_t Z + v \cdot \nabla Z = -\nabla \phi \rho - \rho Z. \quad (11.1)$$

A priori, we only know that $Z(t) \in L^\infty$, just as with $\nabla \psi_j$ of Theorem 2.3. But the term $-\rho Z$, in effect, cancels the term $\nabla \phi \ast \text{div}(\rho Z)$ appearing in Corollary 6.3. Hence,

$$\partial_t Z(t, x) + (v \cdot \nabla Z)(t, x) = \int_{\Omega_t} \nabla^2 \phi(x - y) (Z(t, x) - Z(y, t)) dy, \quad x \in \mathbb{R}^d,$$

which implies that

$$\|Z(t)\|_{C^{1,\alpha}} \leq \|\nabla \phi_0\|_{C^{1,\alpha}} \exp \left( C \int_0^t (1 + \|\nabla v(s)\|_{L^\infty}) ds \right).$$

This gives the $C^{1,\alpha}$ regularity of the patch boundary.

In light of (2.2) and Proposition 8.5, we see that the authors of [3] are using the last vector field in a frame without ever employing the first $d - 1$ vector fields. They are able to do this because they can use the transport properties of $\phi$ to explicitly solve for what we call $Z$ (in terms of the flow map) inside and outside $\Omega$, and ultimately show that it has higher regularity in the normal direction than at first appears. It is critical in their argument, however, that $\phi$ vanish on $\partial \Omega$, thereby eliminating the singularity that would otherwise be in $-\nabla (\phi \rho)$. Their approach could be extended, for example, to nested sums of vortex patches without great difficulty, simply by choosing $\phi_0$ to have a successive series of zeroes at the boundary of each vortex patch. (If would no longer vanish entirely, but it would take on only discrete values, a minor complication.) But this would not work for initial densities whose level sets are $C^{1+\alpha}$ but which have no regularity at all in directions perpendicular to the level sets, which we allow in Theorem 2.3. Any choice of $\phi_0$ that vanishes on these level sets would necessarily vanish on open sets, and hence the level sets would not be regular points of $\phi_0$.

Hence, it appears unavoidable to bring in the idea of pushing forward a frame, then using the $C^{1,\alpha}$ regularity of the vector fields in the tangential direction to obtain the $C^{1,\alpha}$ regularity of $Z$. This is the idea that Chemin [8] brought to bear on the vortex patch problem for the 2D Euler equations.

Another key difference between [3] and this paper is that Picard iteration on parametrizations of the boundary of the patch is used in [3], which avoids the need to regularize the initial data—an issue that was perhaps the most troublesome in our approach. But applying Picard iteration with the initial data in Theorem 2.3 has its own, seemingly insurmountable difficulties, especially in face of the complicated structure of the transport equations in Section 8.
Acknowledgements

H. Bae was supported by NRF-2018R1D1A1B07049015. Work on this paper was accomplished in part while H. Bae was visiting the Department of Mathematics at UC Riverside in 2018, and in part while J. Kelliher was in residence during the Spring 2017 semester at the Institute for Computational and Experimental Research in Mathematics in Providence, RI, which is supported by the National Science Foundation under Grant No. DMS-1439786. The authors would like to thank Joan Verdera for bringing the topic of this paper to their attention.

References


Department of Mathematical Sciences, Ulsan National Institute of Science and Technology (UNIST), Korea
E-mail address: hantaek@unist.ac.kr

Department of Mathematics, University of California, Riverside, USA
E-mail address: kelliher@math.ucr.edu