STREAM FUNCTIONS FOR DIVERGENCE-FREE VECTOR FIELDS

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Abstract. In 1990, Von Wahl and, independently, Borchers and Sohr showed that a divergence-free vector field u in a 3D bounded domain that is tangential to the boundary can be written as the curl of a vector field vanishing on the boundary of the domain. We extend this result to higher dimension and to Lipschitz boundaries in a form suitable for integration in flat space, showing that u can be written as the divergence of an antisymmetric matrix field. We also demonstrate how obtaining a kernel for such a matrix field is dual to obtaining a Biot-Savart kernel for the domain.

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1. Overview

Let u be a divergence-free vector field on a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, that is tangential to the boundary. For a simply connected domain, it is well known that in two dimensions, $u = \nabla^\perp \psi := (-\partial_2 \psi, \partial_1 \psi)$ for a *stream function*, ψ, vanishing on the boundary. It is also well known that in three dimensions, we can write $u = \operatorname{curl} \psi$, where now the *vector potential* ψ is a divergence-free vector field tangential to the boundary. Perhaps somewhat less well-known is that ψ can also be chosen (non-uniquely) to vanish on the boundary, though sacrificing the divergence-free condition. This 3D form of the vector potential was developed in [7, 20], where it is studied in Sobolev, Hölder spaces, for $C^{1,1}$, C^∞ boundaries, respectively.

In higher dimension, we can no longer use a vector field as the potential; instead, we will use an antisymmetric matrix field A vanishing on the boundary, for which $u = \operatorname{div} A$, the divergence applied to A row-by-row. This was the manner it was utilized in [15], without, however, the key antisymmetric condition.

Our main result is Theorem 1.1.

Theorem 1.1. *Let H be the space of divergence-free vector fields on Ω that are tangential to the boundary and that have L^2 coefficients. Let H_c be the closed subspace of* curl*-free vector fields (see (3.1)) in H, let H_0 be its orthogonal complement in H, and let*

$$X_0 := \{A \in H_0^1(\Omega)^{d \times d} \colon A \text{ antisymmetric}\}.$$

Then $H_0 = \operatorname{div} X_0$, and there exists a bounded linear map $S \colon H_0 \to X_0$ with $\operatorname{div} Su = u$.

Specializing to $d = 2, 3$, we can write

$$H_0 = \begin{cases} \nabla^\perp H_0^1(\Omega), & d = 2, \\ \operatorname{curl}_3 H_0^1(\Omega)^3, & d = 3. \end{cases}$$

Because the term *matrix potential* is commonly used in the literature for other purposes, we will adopt the 2D terminology for all dimensions, calling A the *stream function* for u.

Closely connected to stream functions is the Hodge decomposition of L^2-vector fields on Ω. Indeed, one form of the Hodge decomposition in 3D is

$$H = H_c \oplus \operatorname{curl}(H \cap H^1(\Omega)^3).$$

That is, each element of $H_0 := H_c^\perp$ is the image of a classical, divergence-free vector potential tangential to the boundary. Moreover, for any $u \in H_0$, the boundary value problem

$$\begin{cases} \operatorname{curl} \psi = u & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

is (non-uniquely) solvable, and gives the 3D form of the stream function in Theorem 1.1.

In fact, solving the analog of (1.1) in any dimension in the more general setting of an oriented manifold with boundary was worked out by Schwarz in [17]. He shows that for such a manifold with $C^{1,1}$ boundary, given a 1-form α having L^2-regularity and vanishing normal component, the boundary value problem

$$\begin{cases} \delta\beta = \alpha & \text{on } M, \\ \beta|_{\partial M} = 0 & \text{on } \partial M \end{cases}$$

(δ is the codifferential) is solvable for a 2-form having H^1-regularity if and only if

$$\int_M \alpha \wedge *\lambda = 0 \text{ for all } \lambda \in \mathcal{H}^1_N(\Omega).$$

Here, $\mathcal{H}^1_N(\Omega)$ is the space of harmonic fields having vanishing normal component, the analog of H_c, and the integral condition on α defines the analog of H_0.

Schwarz's result is not restricted to 1-forms, but holds for k-forms and also allows non-zero boundary values. It is restricted, however, to $C^{1,1}$ boundaries. For manifolds embedded in $\mathbb{R}^d$, this restriction is loosened in [16], which applies to boundaries even less regular than Lipschitz. The authors show that, given an $(\ell - 1)$-form α for any $0 \le l \le d-1$, there exists an ℓ-form β having prescribed boundary value for which $\delta\beta = \alpha$. They assume, however, that the $(\ell-1)$-st Betti number vanishes. Since we need such a result for $\ell = 2$, this means that the first Betti number must vanish, which means that Ω must be simply connected, an assumption we wish to avoid.

We present our derivation of a stream function here, therefore, because it applies to non-simply connected domains having only a Lipschitz continuous boundary. Moreover, we obtain the stream function non-constructively, using simple functional analytic arguments, avoiding entirely the language of differential forms, making it more accessible and self-contained for our intended primary audience of analysts working in flat space.

Central to our approach is the fact that the divergence operator maps vector fields in $H^1_0(\Omega)^d$ *onto* $L^2_0(\Omega)$, the space of L^2 functions with mean zero. For arbitrary domains, this is a result of Bogovskiǐ [5, 6] (see Lemma 2.5, below). Bogovskiǐ produces an integral kernel for solving the problem $\operatorname{div} u = f$ in a star-shaped domain. This kernel and adaptations of it have been used in other approaches to Theorem 1.1 in 3D, such as [4] for star-shaped domains, but we use Bogovskiǐ's result as a "black box," for with it, we can easily obtain Theorem 1.1 except for the key antisymmetric condition on the stream function.

We assume that Ω is a bounded, connected, open subset of $\mathbb{R}^d$, $d \ge 2$, with Lipschitz boundary, $\partial\Omega$. We define the L^2-based Sobolev spaces, $H^k(\Omega)$ and $H^k_0(\Omega)$, for nonnegative k in the usual way (the boundary is regular enough that all standard definitions are equivalent). Identifying L^2 with its own dual, we also define the dual spaces, $H^{-k}(\Omega) := H^k_0(\Omega)'$.

We will work with the classical function spaces, H and V, of incompressible fluid mechanics:

$$H := \{u \in L^2(\Omega)^d \colon \operatorname{div} u = 0, \, u \cdot \boldsymbol{n} = 0\},$$

$$V := \{u \in H_0^1(\Omega)^d \colon \operatorname{div} u = 0\}.$$

The divergence here is defined in terms of weak derivatives, and $u \cdot \boldsymbol{n}$ is defined as an element of $H^{-\frac{1}{2}}(\partial\Omega)$ in terms of a trace (see Lemma 2.2), $\boldsymbol{n}$ being the outward unit normal vector. Both H and V are Hilbert spaces with norms and inner products as subspaces of L^2 and H_0^1. By virtue of the Poincaré inequality, we can use

$$\begin{aligned} (f,g)_{H_0^1} &:= (\nabla f, \nabla g)_{L^2}, \quad \|f\|_{H_0^1} := \|\nabla f\|_{L^2}, \\ (u,v)_V &:= (\nabla u, \nabla v)_{L^2}, \quad \|u\|_V := \|\nabla u\|_{L^2}. \end{aligned}$$

With these very cursory definitions out of the way, we give in Section 2 some further necessary background material. In Section 3, we prove our main result, Theorem 1.1, extending it to the space V in Section 4. In Section 5 we show how the classical 3D vector potentials can be obtained from the stream function of Theorem 1.1.

In Section 6 we demonstrate that the Biot-Savart law, which recovers a vector field in H_0 from its vorticity (curl), is, in a precise way, dual to the problem of obtaining a stream function from a velocity field in H_0. We show that if there is an integral kernel associated with one of these problems it is also the kernel associated with the other problem.

Throughout, we follow the convention that $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$ or $\|\cdot\|_H$.

We write (u,v) for the inner product in L^2 or H. We write v^i for the i-th coordinate of a vector v; A^i_j for the element in the i-th row, j-th column of a matrix A; A^i for the i-th row of A; A_j for the j-th column of A. We follow the convention that repeated indices are implicitly summed, even when both indices are superscripts or both are subscripts.

2. Background material

We briefly present some necessary background results.

Definition 2.1. *As in [19], we define the space*

$$E(\Omega) := \{u \in L^2(\Omega)^d \colon \operatorname{div} u \in L^2(\Omega)\},$$

endowed with the norm, $\|u\| + \|\operatorname{div} u\|$.

We frequently integrate by parts using Lemma 2.2 (see Theorem 2.5 and (2.17) of [13]):

Lemma 2.2. *There exists a normal trace operator from $E(\Omega)$ to $H^{-1/2}(\partial\Omega)$ that continuously extends $u \mapsto u \cdot \boldsymbol{n}|_{\partial\Omega}$ from $C(\overline{\Omega})$ to $E(\Omega)$. We will simply write $u \cdot \boldsymbol{n}$ rather than naming this trace operator. For all $u \in E(\Omega)$, $\varphi \in H^1(\Omega)$,*

$$(u, \nabla\varphi) = -(\operatorname{div} u, \varphi) + \int_{\partial\Omega} (u \cdot \boldsymbol{n})\varphi,$$

where we have written $(u \cdot \boldsymbol{n}, \varphi)_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}$ in the form of a boundary integral.

Poincaré's inequality holds not just for V, but for the larger space $H \cap H^1(\Omega)^d$:

Lemma 2.3. *There exists a constant $C = C(\Omega)$ such that for all $u \in H \cap H^1(\Omega)^d$,*

$$\|u\| \le C \|\nabla u\|.$$

Proof. For any $u \in H$,

$$\int_\Omega u^j = \int_\Omega u \cdot \nabla x^j = -\int_\Omega \operatorname{div} u \, x^j + \int_{\partial\Omega} (u \cdot \boldsymbol{n})\, x^j = 0.$$

Hence, u has mean value zero, so Poincaré's inequality holds in the form stated. □

Key tools for us will be the decomposition of vector fields in $H_0^1(\Omega)$ given in Proposition 2.4 and the surjectivity of the divergence operator in Lemma 2.5. These results employ the space

$$L_0^2(\Omega) := \{f \in L^2(\Omega) \colon \int_\Omega f = 0\}.$$

Proposition 2.4. *The orthogonal decomposition, $H_0^1(\Omega)^d = V \oplus V^\perp$, holds with*

$$V^\perp = \{z \in H_0^1(\Omega)^d \colon \Delta z = \nabla q \text{ for some } q \in L_0^2(\Omega)\},$$

and $\|P_{V^\perp}\varphi\| \le C\|\operatorname{div}\varphi\|$.

Proof. This decomposition is given in Corollary 2.3 p. 23 of [13] (also see Lemma 2.2 of [14]). The bound $\|P_{V^\perp}\varphi\| \le C\|\operatorname{div}\varphi\|$ follows, for instance, from the Stokes problem bound in Exercise IV.1.1 of [11]. □

Lemma 2.5. *[Bogovskiĭ [5, 6]] For any $f \in L_0^2(\Omega)$ there exists $v \in H_0^1(\Omega)^d$ for which $\operatorname{div} v = f$. We can choose the (non-unique) solutions in such a way as to define a bounded linear operator $R\colon L_0^2(\Omega) \to H_0^1(\Omega)^d$ for which $\|\nabla Rf\| \le C\|f\|$. Moreover, we can assume that R maps into the space $V^\perp$.*

Proof. For the proof of all but the last sentence, see Bogovskiĭ [5, 6] or Theorem 2.4 of [7]. Then, for any $f \in L_0^2(\Omega)$, $\operatorname{div}(P_{V^\perp}Rf) = \operatorname{div} Rf = f$ and

$$\|\nabla(P_{V^\perp}Rf)\| = \|P_{V^\perp}Rf\|_{H_0^1(\Omega)^d} \le \|Rf\|_{H_0^1(\Omega)^d} = \|\nabla Rf\|.$$

So because $P_{V^\perp}$ is a continuous linear operator, we can replace R by $P_{V^\perp}R$. □

In fact, Bogovskiĭ in [5, 6] showed that the divergence is surjective for an arbitrary domain in $\mathbb{R}^d$. See, for instance, the historical comments on pages 208-209 of [2].

The difficult part of proving Lemma 2.5 is obtaining the surjectivity of the divergence as a map from $H_0^1(\Omega)^d$ to $L_0^2(\Omega)$: once that is obtained (or even just that the range of div is closed), the bounded linear (partial) inverse map R follows from basic functional analysis, by arguing much as we do in the proof of Theorem 1.1 in Section 3. (And see Remark 3.6.)

Moreover, since $P_{V^\perp}$ does not change the divergence of a vector field, the constant in the inequality in Lemma 2.5 is at least as small as the constant in Proposition 2.4. (This is a little misleading, however, as Lemma 2.5 is generally used to prove the estimates on the Stokes problem that lead to the inequality in Proposition 2.4.)

From R of Lemma 2.5, we define a matrix-valued operator, which we continue to call R, by applying R on each component of any vector in $L_0^2(\Omega)^d$:

$$R\colon L_0^2(\Omega)^d \to H_0^1(\Omega)^{d\times d}, \quad (Ru)^i := Ru^i. \tag{2.1}$$

3. Proof of main result

In this section we prove our main result, Theorem 1.1. We present first some imporant existing results then establish a series of lemmas and propositions we will use in the (short) body of the proof of Theorem 1.1, with which we close the section.

Define the subspace

$$H_c := \{u \in H \colon \operatorname{curl} u = 0\}$$

of H. Here, we use the curl operator on $\mathbb{R}^d$ in the form,

$$\operatorname{curl} u := \nabla u - (\nabla u)^T. \tag{3.1}$$

That is, $\operatorname{curl} u$ is twice the antisymmetric gradient, the $d \times d$ matrix-valued function with $(\operatorname{curl} u)^i_j = \partial_j u^i - \partial_i u^j$. This form of the curl is convenient for integrating by parts (applying the divergence theorem) in flat space. In 2D, we can define $\operatorname{curl} u := \partial_1 u^2 - \partial_2 u^1$, the scalar curl, and in 3D we can define it as a vector in the usual way, denoting it curl_3 for clarity.

H_c is clearly closed, so we can define

$$H_0 := H_c^\perp,$$

the orthogonal complement of H_c in H. Hence, $H = H_0 \oplus H_c$.

Remark 3.1. *H_c is finite-dimensional for a large class of domains for which $\partial\Omega$ has a finite number of components. For smooth boundaries, this follows, for instance, from the discussion in Section 4.1 of* [12]*. For special classes of 3D Lipschitz domains, Helmholtz domains of* [3]*, H_c (and H_0) can be characterized by making "cuts" in Ω that leave the remaining domain simply connected. This idea goes back to Helmholtz; see the historical comments in* [9]*.*

In [15] (Corollary 7.5), the simple tool in Lemma 3.2 was used to investigate conditions under which solutions to the Navier-Stokes equation for incompressible fluids converge to a solution to the Euler equations (the so-called *vanishing viscosity limit*).

Lemma 3.2. *For any $u \in H$ there exists (a non-unique) $A \in H_0^1(\Omega)^{d\times d}$ such that $u = \operatorname{div} A$; that is, such that $u^i = \partial_j A^i_j$.*

The idea of the proof is that a simple integration by parts as in the proof of Lemma 2.3 shows that each component of any $v \in H$ lies in $L^2_0(\Omega)$. But by Lemma 2.5, div maps $H^1_0(\Omega)^d$ onto $L^2_0(\Omega)$, so we can obtain each row of A independently. The proof of Lemma 3.2 is therefore quite simple, but it relies on the powerful and deep result in Lemma 2.5.

Left open in [15] was whether it could be assured that A in Lemma 3.2 is antisymmetric. In fact, such antisymmetry can be obtained, and was obtained in 3D by Borchers and Sohr in Theorem 2.1, Corollary 2.2 of [7], whose lowest regularity result can be stated as follows:

Lemma 3.3. *Assume that $d = 3$ and $\partial\Omega$ is $C^{1,1}$. For any $u \in H_0$ there exists $v \in H^1_0(\Omega)^3$ such that $u = \operatorname{curl}_3 v$ and $\Delta \operatorname{div} v = 0$. Moreover, one can choose the solutions in such a way as to define a bounded linear operator $S\colon H_0 \to H^1_0(\Omega)^3$ with $\|\nabla S u\| \le C\|u\|$.*

To see that Lemma 3.3 provides a 3D form of an extension of Lemma 3.2 to antisymmetric matrices, note that any 3×3 antisymmetric matrix can be written in the form,

$$A = \begin{pmatrix} 0 & \psi^3 & -\psi^2 \\ -\psi^3 & 0 & \psi^1 \\ \psi^2 & -\psi^1 & 0 \end{pmatrix}. \tag{3.2}$$

We can define a bijection Q from a vector in $\mathbb{R}^3$ to an antisymmetric $d\times d$ matrix, by setting $Q(\psi) = Q(\psi^1, \psi^2, \psi^3)$ to be the matrix in (3.2), and we can write that $\operatorname{div} Q\psi = \operatorname{curl}_3 \psi$. The claim in Theorem 1.1, then, is the natural extension of Lemma 3.3 to $d \ge 2$.

The simple argument in Proposition 3.4 shows that $\operatorname{div} X_0$ is at least dense in H_0:

Proposition 3.4. *$H_0 = \overline{\operatorname{div} X_0}$.*

Proof. First, we show that $\operatorname{div} X_0$ is a subspace of H. To see this, observe that if $u \in \operatorname{div} X_0$ then $u^i = \operatorname{div} A^i = \partial_j A^i_j$. Hence, $\operatorname{div} u = \partial_{ij} A^i_j = -\partial_{ij} A^j_i = -\partial_{ji} A^i_j = -\partial_{ij} A^i_j = -\operatorname{div} u$, so $\operatorname{div} u = 0$. (That $\operatorname{div} u = \operatorname{div}\operatorname{div} A = 0$ is a reflection of $\delta^2 = 0$ when A is expressed as a 2-form.)

Moreover, since A^i_j is constant along the boundary, ∇A^i_j is normal to the boundary, so we can write, $\nabla A^i_j = \alpha^i_j \boldsymbol{n}$, where

$$\alpha^i_j = \frac{\partial A^i_j}{\partial \boldsymbol{n}} = -\frac{\partial A^j_i}{\partial \boldsymbol{n}} = -\alpha^j_i.$$

Then,

$$\partial_j A^i_j = \nabla A^i_j \cdot \mathbf{e}^j = \alpha^i_j \boldsymbol{n} \cdot \mathbf{e}^j = \alpha^i_j n^j$$

so, using that $\alpha^i_j = -\alpha^j_i$,

$$u \cdot \boldsymbol{n} = \operatorname{div} A \cdot \boldsymbol{n} = \operatorname{div} A^i n^i = \partial_j A^i_j n^i = \alpha^i_j n^j n^i = -\alpha^j_i n^j n^i = -\alpha^i_j n^j n^i = -u \cdot \boldsymbol{n},$$

so $u \cdot \boldsymbol{n} = 0$. We conclude that $\operatorname{div} X_0 \subseteq H$.

We now show that $(\operatorname{div} X_0)^\perp = H_c$. Let $A \in X_0$ and $v \in H$ be arbitrary. Then $u := \operatorname{div} A$ is an arbitrary element of $\operatorname{div} X_0$. Applying Lemma 2.2 and using $A = 0$ on $\partial\Omega$,

$$\begin{aligned}(u, v) &= (\operatorname{div} A, v) = -(A, \nabla v) = -(A, \nabla v - (\nabla v)^T) - (A, (\nabla v)^T) \\ &= -(A, \operatorname{curl} v) - (A^T, \nabla v) = -(A, \operatorname{curl} v) + (A, \nabla v).\end{aligned}$$

Hence, $(A, \nabla v) = (1/2)(A, \operatorname{curl} v)$, and because both A and $\operatorname{curl} v$ are antisymmetric,

$$(u, v) = -(A, \nabla v) = -\frac{1}{2}(A, \operatorname{curl} v) = -\sum_{i<j} A^i_j (\operatorname{curl} v)^i_j.$$

We can choose the components A^i_j independently for $i < j$, and $H^1_0(\Omega)$ is dense in $L^2(\Omega)$, so we conclude that $(u, v) = 0$ for all $u \in \operatorname{div} X_0$ if and only if $\operatorname{curl} v = 0$; that is, if and only if $v \in H_c$. It then follows that $(\operatorname{div} X_0)^\perp = H_c$ so that, in fact, $\overline{\operatorname{div} X_0} = ((\operatorname{div} X_0)^\perp)^\perp = H_c^\perp = H_0$. $\square$

The operator R of (2.1) allows us to easily establish that $\operatorname{div} X_0$ actually yields all of H_0:

Proposition 3.5. *$H_0 = \operatorname{div} X_0$.*

Proof. We have, $\operatorname{div} X_0 = \operatorname{div}(R \operatorname{div} X_0) = \operatorname{div} Y$, where $Y = R \operatorname{div} X_0$. It follows from Proposition 3.4 that $\operatorname{div} Y$ is dense in H_0. If we can show that it is closed, then we are done.

Let (u_n) be a sequence in $\operatorname{div} Y$ converging to u in H_0. Then $u_n = \operatorname{div} B_n$ with $B_n = Ru_n$ in Y, and we have from Lemma 2.5 that $\|\nabla B_n\| \le C \|u_n\|$. Since (u_n) converges, it is Cauchy and hence (B_n) is Cauchy and so converges to some $B \in Y$ with $u = \operatorname{div} B$. This shows that $H_0 = \operatorname{div} Y = \operatorname{div} X_0$. $\square$

It remains only to obtain the bounded linear map S of Theorem 1.1. Examining the proof of Proposition 3.5, we see that $B_n = Ru_n$ in Y has some D_n in X_0 for which $R \operatorname{div} D_n = B_n$, but the convergence of (B_n) does not mean the convergence of (D_n). To surmount this difficulty, and obtain S, we restrict the domain of div to a subspace:

Proof of Theorem 1.1. Observe that $\operatorname{div} A = \operatorname{div} B$ for $A, B \in X_0$ if and only if $B = A + E$ for some E in $V^d \cap X_0$, a closed subspace of X_0. Letting $Y_0 = (V^d \cap X_0)^\perp$, the orthogonal complement of $V^d \cap X_0$ in X_0 *as a Hilbert space*, $\operatorname{div} \colon Y_0 \to H_0$ is a continuous bijection. It follows from a corollary of the open mapping theorem (see, for instance, Corollary 2.7 of [8]) that the inverse map, $S := \operatorname{div}|_{Y_0}^{-1}$, is also continuous. But this means that, $\|Su\|_{X_0} = \|Su\|_{Y_0} \le C \|u\|_{H_0}$, giving us the bounded linear map of Theorem 1.1. $\square$

The Baire category theorem appears through the proof of the corollary to the open mapping theorem we applied. Hence, the constant we obtain in $\|\nabla Su\| \le C \|\operatorname{div} u\|$ is not effectively computable, although we can see that C is no smaller than the constant in Lemma 2.5.

Remark 3.6. *Although the adjoints to the two forms of* div *appearing in Lemma 2.5 and Theorem 1.1 never appear explicitly, they are, in a sense, hiding in the proofs. It can be shown that the adjoint of* $\operatorname{div} \colon X_0 \to H_0$ *is* $-(1/2) \operatorname{curl}$*, whose null space is* H_c*. Since* div *is a closed map,* $\operatorname{div} X_0$ *is closed if and only if it equals* $H_c^\perp =: H_0$*. Similarly, it can be shown that the adjoint of* $\operatorname{div} \colon H^1_0(\Omega)^d \to L^2_0(\Omega)$ *is* $-\nabla$*, whose null space is trivial. Hence,* $\operatorname{div} H^1_0(\Omega)^d$ *is closed if only if it equals all of* $L^2_0(\mathbb{R}^d)$*. Proving that the range of either version of* div *is closed is the hard part of each proof, but we were able to leverage the powerful result in Lemma 2.5 to obtain the hard part for Theorem 1.1 with minimal effort.*

We avoided characterizing the space $Y_0 = (V^d \cap X_0)^\perp$ explicitly, but given that the adjoint of $\operatorname{div}\colon X_0 \to H_0$ *is* $-(1/2)\operatorname{curl}$, *one can show that* $Y_0 = \{z \in X_0 \colon \Delta z = \operatorname{curl} q \text{ for some } q \in L^2_0(\Omega)^d\}$, *in analogy with Proposition 2.4. In 3D, this is* $Y_0 = \{z \in H^1_0(\Omega)^3 \colon \Delta z = \operatorname{curl}_3 q, q \in L^2_0(\Omega)^d\}$, *which yields* $\Delta \operatorname{div} Su = 0$, *as in Lemma 3.3.*

4. Higher regularity

Bogovskiǐ in [5, 6] showed more than what we stated in Lemma 2.5 (see Theorem 2.4 of [7]):

Lemma 4.1. *[Bogovskiǐ [5, 6]] Let $p \in (1, \infty)$ and $m \geq 0$ be an integer. Define $H^{m,p}_{0,0}(\Omega)$ to be the functions in $H^{m,p}_0(\Omega)$ having mean zero. There exists a bounded linear operator $R = R_{m,p}\colon H^{m,p}_{0,0}(\Omega) \to H^{m+1,p}_0(\Omega)^d$ satisfying* $\operatorname{div} Rf = f$ *with* $\left\|\nabla^{m+1} Rf\right\|_{L^p(\Omega)} \leq C \left\|\nabla^m f\right\|_{L^p(\Omega)}$.

Restricting ourselves to $p = 2$, we define, as in (2.1), a matrix-valued operator $R_m = R_{m,2}$:

$$R_m\colon H^m_0(\Omega)^d \to H^{m+1}_0(\Omega)^{d\times d}, \quad (R_m u)^i := R_m u^i.$$

We will use Lemma 4.1 to study the stream function for an element of V.

Theorem 4.2. *The map S of Theorem 1.1 also maps $V \cap H_0$ continuously onto $Y_0 \cap H^2_0(\Omega)^{d\times d}$, where $Y_0 = (V^d \cap X_0)^\perp$.*

Proof. The space $Y^2_0 := Y_0 \cap H^2_0(\Omega)^{d\times d}$ is dense in Y_0 and $\operatorname{div}\colon Y_0 \to H_0$ is a continuous surjection, so $\operatorname{div} Y^2_0$ is dense in H_0. Moreover, $\operatorname{div} Y^2_0 \subseteq V \cap H_0$, so $\operatorname{div} Y^2_0$ is dense in $V \cap H_0$. Then, arguing as in the proof of Proposition 3.5, $\operatorname{div} Y^2_0 = \operatorname{div}(R_1 \operatorname{div} Y^2_0)$ is closed in $V \cap H_0$ and hence $\operatorname{div} Y^2_0 = V \cap H_0$. Because $\operatorname{div}|_{Y_0}$ is injective it also holds that $\operatorname{div}|_{Y^2_0}$ is injective. Finally, arguing as in the proof of Theorem 1.1, the inverse map, $\operatorname{div}|^{-1}_{Y^2_0}$, is continuous. But this is the same map S as in Theorem 1.1, restricted to $V \cap H_0$. □

Remark 4.3. *Using R_m, one can extend Theorem 4.2 to $S\colon H_0 \cap H^m_0(\Omega)^d \to Y_0 \cap H^{m+1}_0(\Omega)^{d\times d}$, though its utility is likely limited for $m \geq 2$. Similarly, one can employ Lemma 4.1 to develop L^p bounds in analog with Theorem 1.1.*

5. 3D vector potentials

We can use Theorem 1.1 to obtain the more classical versions of 3D stream functions or vector potentials of Propositions 5.1 and 5.2 (cf., Theorems 3.5 and 3.6 Chapter I of [13] or Theorem 3.12 and 3.17 of [1]).

Proposition 5.1. *Let $u \in H_0$ for $d = 3$. There exists a vector potential $\overline{\psi} \in H$ for which* $\operatorname{curl}_3 \overline{\psi} = u$. *The vector potential is unique up to the addition of an arbitrary element in H_c; or, equivalently, the vector potential is unique if we require it to lie in H_0. If $\partial\Omega$ is $C^{1,1}$ then $\overline{\psi} \in H \cap H^1(\Omega)^3$.*

Proof. First, we show existence. Let ψ be the 3D stream function given by Theorem 1.1 and let p be the unique (up to an additive constant) solution to the Neumann problem,

$$\begin{cases} \Delta p = -\operatorname{div} \psi & \text{in } \Omega, \\ \nabla p \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

If $\partial\Omega$ is Lipschitz, we can only conclude that $p \in H^1(\Omega)$ so $\nabla p \in L^2(\Omega)^3$, but if $\partial\Omega$ is $C^{1,1}$ then $p \in H^2(\Omega)$ so $\nabla p \in H^1(\Omega)^3$. Letting $\overline{\psi} = \psi + \nabla p$, we see that

$$\begin{cases} \operatorname{curl}_3 \overline{\psi} = u & \text{in } \Omega, \\ \operatorname{div} \overline{\psi} = 0 & \text{in } \Omega, \\ \overline{\psi} \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

Hence, $\overline{\psi} \in H$ with $\operatorname{curl}_3 \overline{\psi} = u$, as required, with $\overline{\psi} \in H \cap H^1(\Omega)^3$ if $\partial\Omega$ is $C^{1,1}$.

Adding any element of H_c to $\overline{\psi}$ clearly yields another vector potential for u, and the difference of any two vector potentials for u lies in H and is curl-free; that is, it lies in H_c. This proves the uniqueness statement. $\square$

Define the space,

$$\widetilde{H} := \{\psi \in L^2(\Omega)^3 \colon \operatorname{div} \psi = 0, \operatorname{curl} \psi \in L^2(\Omega)^3, \psi \times \boldsymbol{n} = 0 \text{ on } \partial\Omega\}$$

with the norm $\|\psi\|_{\widetilde{H}} := \|\psi\| + \|\operatorname{curl} \psi\|$. That $\psi \times \boldsymbol{n}$ makes sense in terms of a trace is shown in Theorem 2.11 of [13]. Also let

$$\widetilde{H}_c := \{\psi \in \widetilde{H} \colon \operatorname{curl} \psi = 0\}.$$

Proposition 5.2. *Let $u \in H_0$ for $d = 3$. There exists a vector potential $\overline{\psi} \in \widetilde{H}$ for which $\operatorname{curl}_3 \overline{\psi} = u$. The vector potential is unique up to the addition of an arbitrary element in $\widetilde{H}_c$. If $\partial\Omega$ is $C^{1,1}$ then $\overline{\psi} \in \widetilde{H} \cap H^1(\Omega)^3$.*

Proof. The proof is the same as that of Proposition 5.1, but using the boundary condition $p = 0$ on $\partial\Omega$ in (5.1), noting that then $\nabla p \times \boldsymbol{n} = 0$. As in (5.2), this gives $\operatorname{curl}_3 \overline{\psi} = u$ and $\operatorname{div} \overline{\psi} = 0$ but with $\overline{\psi} \times \boldsymbol{n} = \psi \times \boldsymbol{n} + \nabla p \times \boldsymbol{n} = 0$ on $\partial\Omega$. Adding any element of $\widetilde{H}_c$ to $\overline{\psi}$ clearly yields another vector potential for u, and the difference of any two vector potentials for u lies in $\widetilde{H}$ and is curl-free; that is, it lies in $\widetilde{H}_c$. This proves the uniqueness statement. $\square$

Suppose that $\Omega \subseteq \mathbb{R}^3$ has a finite number of boundary components $\Gamma_0, \cdots, \Gamma_N$. Then the vector potential $\overline{\psi}$ of Proposition 5.2 is unique if one imposes the condition $\int_{\Gamma_i} \overline{\psi} \cdot \boldsymbol{n} = 0$ for all i. This is shown in Theorem 3.6 Chapter I of [13] and 3.17 of [1]. The idea, in essence, is to use the boundary condition $p = c_i$ on Γ_i instead of $p = 0$ on $\partial\Omega$ in (5.1), and show that, fixing $c_0 = 0$, there exists a unique choice of the c_i such that $\int_{\Gamma_i} \nabla p \cdot \boldsymbol{n} = -\int_{\Gamma_i} \psi \cdot \boldsymbol{n}$ for all i. See, for instance, the argument on pages 49-50 of [13].

Remark 5.3. *The boundary condition $\psi \times \boldsymbol{n} = 0$ in the definition of $\widetilde{H}$ corresponds to $A\boldsymbol{n} = 0$ via the bijection given by (3.2). This suggests that Proposition 5.2 has a natural higher-dimensional formulation. Indeed for smooth boundaries it does, as follows from Theorem 3.1.1 of [18], in which $\overline{\psi}$ becomes a co-closed 2-form.*

6. A Biot-Savart kernel?

The Biot-Savart law is the classical method for obtaining a vector field in, say $H_0 \cap H^1(\Omega)^d$, having a given vorticity in $L^2(\Omega)$. But the existence of an integral representation for this law, that is, of a Biot-Savart kernel, for a bounded domain is a largely open question: the existence for all of $\mathbb{R}^d$ and for a bounded domain in $\mathbb{R}^2$ is quite classical, but only recently, in [10], has a kernel for a 3D bounded domain been obtained, and that was for domains with smooth boundary. In dimensions higher than 3 a kernel has not been obtained even for smooth domains. (Also, see the introductory comments in [10].)

We can show, however, the conditional result in Theorem 6.1: a Biot-Savart kernel exists if and only if a kernel for the stream function exists, and there is a duality between them.

Theorem 6.1. *We say that $K \in L^1(\Omega^2)^d$ is a kernel for the Biot-Savart law on Ω if for all antisymmetric $B \in C(\overline{\Omega})^{d\times d}$,*

$$u^i(x) = \int_\Omega K^j(x,y) B^i_j(y)\, dy \tag{6.1}$$

lies in H_0 with $\operatorname{curl} u = B$. We say that $T \in L^1(\Omega^2)^d$ is a kernel for the stream function on Ω if for all $v \in H_0 \cap C^\infty(\overline{\Omega})^d$,

$$A^i_j(y) = \int_\Omega T_j(x,y) v^i(x)\, dx - \int_\Omega T_i(x,y) v^j(x)\, dx \tag{6.2}$$

lies in X_0 with $\operatorname{div} A = v$. A kernel K exists if and only if a kernel T exists, and in such a case, we can set $K = T$.

Proof. Assume that T exists. Let $v \in H_0 \cap C^\infty(\overline{\Omega})^d$ and let A be as given in (6.2). Let $u \in H_0 \cap C^\infty(\overline{\Omega})^d$ with $\operatorname{curl} u = B$. Then, applying Fubini's theorem,

$$\begin{aligned}
(2u, v) &= 2(u, \operatorname{div} A) = -2(\nabla u, A) = -(\nabla u, A) - ((\nabla u)^T, A^T) \\
&= -(\nabla u, A) + ((\nabla u)^T, A) = -(\operatorname{curl} u, A) = -(B, A) \\
&= \int_\Omega \int_\Omega B^i_j(y) \left[T_i(x,y) v^j(x)\, dx - \int_\Omega T_j(x,y) v^i(x)\, dx \right] dy \\
&= \int_\Omega \int_\Omega B^i_j(y) T_i(x,y) v^j(x)\, dx - \int_\Omega \int_\Omega B^i_j(y) T_j(x,y) v^i(x)\, dx\, dy \\
&= \int_\Omega \int_\Omega B^i_j(y) T_i(x,y) v^j(x)\, dx - \int_\Omega \int_\Omega B^j_i(y) T_i(x,y) v^j(x)\, dx\, dy \\
&= 2\int_\Omega \int_\Omega B^i_j(y) T_i(x,y) v^j(x)\, dx\, dy = (2w, v),
\end{aligned}$$

where

$$w(x) = \int_\Omega T_i(x,y) B^i_j(y)\, dy.$$

Since $H_0 \cap C^\infty(\overline{\Omega})^d$ is dense in H_0 it follows that we must have $u = w$. Examining (6.1), then, we see that we can set $K = T$.

To show that the existence of K implies the existence of T, we reverse the order of the integrations by parts. □

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References

[1] C. Amrouche, C. Bernardi, M. Dauge, and V. Girault. Vector potentials in three-dimensional non-smooth domains. *Math. Methods Appl. Sci.*, 21(9):823–864, 1998. 7, 8

[2] Chérif Amrouche, Philippe G. Ciarlet, and Cristinel Mardare. On a lemma of Jacques-Louis Lions and its relation to other fundamental results. *J. Math. Pures Appl. (9)*, 104(2):207–226, 2015. 4

[3] R. Benedetti, R. Frigerio, and R. Ghiloni. The topology of Helmholtz domains. *Expo. Math.*, 30(4):319–375, 2012. 5

[4] Luigi C. Berselli and Placido Longo. Classical solutions for the system $\operatorname{curl} v = g$, with vanishing Dirichlet boundary conditions. *Discrete Contin. Dyn. Syst. Ser. S*, 12(2):215–229, 2019. 2

[5] M. E. Bogovskiĭ. Solution of the first boundary value problem for an equation of continuity of an incompressible medium. *Dokl. Akad. Nauk SSSR*, 248(5):1037–1040, 1979. 2, 4, 7

[6] M. E. Bogovskiĭ. Solutions of some problems of vector analysis, associated with the operators div and grad. In *Theory of cubature formulas and the application of functional analysis to problems of mathematical physics*, volume 1980 of *Trudy Sem. S. L. Soboleva, No. 1*, pages 5–40, 149. Akad. Nauk SSSR Sibirsk. Otdel., Inst. Mat., Novosibirsk, 1980. 2, 4, 7

[7] Wolfgang Borchers and Hermann Sohr. On the equations rot $v = g$ and div $u = f$ with zero boundary conditions. *Hokkaido Math. J.*, 19(1):67–87, 1990. 1, 4, 5, 7

[8] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011. 6

[9] Jason Cantarella, Dennis DeTurck, and Herman Gluck. Vector calculus and the topology of domains in 3-space. *Amer. Math. Monthly*, 109(5):409–442, 2002. 5

[10] Alberto Enciso, M. Ángeles García-Ferrero, and Daniel Peralta-Salas. The Biot-Savart operator of a bounded domain. *J. Math. Pures Appl. (9)*, 119:85–113, 2018. 8

[11] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations*. Springer Monographs in Mathematics. Springer, New York, second edition, 2011. Steady-state problems. 4

[12] Peter B. Gilkey. *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*, volume 11 of *Mathematics Lecture Series*. Publish or Perish, Inc., Wilmington, DE, 1984. 5

[13] Vivette Girault and Pierre-Arnaud Raviart. *Finite element methods for Navier-Stokes equations*, volume 5 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1986. Theory and algorithms. 3, 4, 7, 8

[14] Mihaela Ignatova, Gautam Iyer, James P. Kelliher, Robert L. Pego, and Arghir D. Zarnescu. Global existence for two extended Navier-Stokes systems. *Commun. Math. Sci.*, 13(1):249–267, 2015. 4

[15] James P. Kelliher. Vanishing viscosity and the accumulation of vorticity on the boundary. *Communications in Mathematical Sciences*, 6(4):869–880, 2008. 1, 5

[16] Dorina Mitrea, Marius Mitrea, and Sylvie Monniaux. The Poisson problem for the exterior derivative operator with Dirichlet boundary condition in nonsmooth domains. *Commun. Pure Appl. Anal.*, 7(6):1295–1333, 2008. 2

[17] G. Schwarz. The existence of solutions of a general boundary value problem for the divergence. *Math. Methods Appl. Sci.*, 17(2):95–105, 1994. 2

[18] Günter Schwarz. *Hodge decomposition—a method for solving boundary value problems*, volume 1607 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1995. 8

[19] Roger Temam. *Navier-Stokes equations*. AMS Chelsea Publishing, Providence, RI, 2001. Theory and numerical analysis, Reprint of the 1984 edition. 3

[20] Wolf von Wahl. On necessary and sufficient conditions for the solvability of the equations rot $u = \gamma$ and div $u = \varepsilon$ with u vanishing on the boundary. In *The Navier-Stokes equations (Oberwolfach, 1988)*, volume 1431 of *Lecture Notes in Math.*, pages 152–157. Springer, Berlin, 1990. 1

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