J. fixed point theory appl. *Online First*

DOI 10.1007/s11784-008-0062-9

Journal of Fixed Point Theory and Applications

Nonarchimedean Cantor set and string

Michel L. Lapidus and Hùng Lũ'

Dedicated to Vladimir Arnold, on the occasion of his jubilee

Abstract. We construct a nonarchimedean (or p-adic) analogue of the classical ternary Cantor set $\mathcal{C}$. In particular, we show that this nonarchimedean Cantor set $\mathcal{C}_3$ is self-similar. Furthermore, we characterize $\mathcal{C}_3$ as the subset of 3-adic integers whose elements contain only 0's and 2's in their 3-adic expansions and prove that $\mathcal{C}_3$ is naturally homeomorphic to $\mathcal{C}$. Finally, from the point of view of the theory of fractal strings and their complex fractal dimensions [7, 8], the corresponding nonarchimedean Cantor string resembles the standard archimedean (or *real*) Cantor string perfectly.

Mathematics Subject Classification (2000). Primary 11M41, 26E30, 28E30, 28A12, 26A80; Secondary 11M36, 12J25, 28A75, 28A78.

Keywords. Fractal geometry, p-adic analysis, zeta functions, complex fractal dimensions, nonarchimedean Cantor set and string.

1. Introduction

Our goal in this article is to provide a good *nonarchimedean* (or *p-adic*) analogue of the classical Cantor ternary set $\mathcal{C}$ and to show that it satisfies a counterpart of some of the key properties of $\mathcal{C}$ in this nonarchimedean context. We also show that the corresponding p-adic fractal string, called the nonarchimedean Cantor string and denoted by $\mathcal{CS}_3$, is an exact analogue of the ordinary archimedean Cantor string, a central example in the theory of real (or archimedean) fractal strings and their complex dimensions [7, 8]. Furthermore, we compute the geometric zeta function of $\mathcal{CS}_3$ and the associated complex fractal dimensions.

In a forthcoming paper [9], we will develop a general framework for formulating a theory of self-similar p-adic (or nonarchimedean) strings and their complex fractal dimensions. Besides answering a natural mathematical question, these results may be useful in various aspects of mathematical physics, including p-adic quantum mechanics and string theory, where extensions from the archimedean to the nonarchimedean setting have been extensively explored [12].

1.1. p-adic numbers

Let $p \in \mathbb{N}$ be a fixed prime number. For any nonzero $x \in \mathbb{Q}$, we can always write $x = p^v \cdot a/b$ with $a, b \in \mathbb{Z}$ and some unique $v \in \mathbb{Z}$ so that p does not divide ab. The *p-adic norm* is a function $|\cdot|_p : \mathbb{Q} \to [0, \infty)$ given by

$$|x|_p = p^{-v} \quad \text{and} \quad |0|_p = 0.$$

One can verify that $|\cdot|_p$ is indeed a norm on $\mathbb{Q}$. Furthermore, it satisfies a *strong triangle inequality*: for any $x, y \in \mathbb{Q}$, we have $|x + y|_p \leq \max\{|x|_p, |y|_p\}$; the induced metric is therefore called an *ultrametric*. This inequality is called the *nonarchimedean property* because for each $x \in \mathbb{Q}$, $|nx|_p$ will never exceed $|x|_p$ for any $n \in \mathbb{N}$. The metric completion of $\mathbb{Q}$ with respect to the p-adic norm is the field $\mathbb{Q}_p$ of p-adic numbers. More concretely, there is a unique representation of every $z \in \mathbb{Q}_p$: $z = a_v p^v + \cdots + a_0 + a_1 p + a_2 p^2 + \cdots$ for some $v \in \mathbb{Z}$ and $a_i \in \{0, 1, \ldots, p-1\}$ for all $i \geq v$. An important subspace of $\mathbb{Q}_p$ is the unit ball, $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$, which can also be represented as follows:

$$\mathbb{Z}_p = \{a_0 + a_1 p + a_2 p^2 + \cdots \mid a_i \in \{0, 1, \ldots, p-1\}, \forall i \geq 0\}.$$

Using this *p-adic expansion*, we can see that

$$\mathbb{Z}_p = \bigcup_{c=0}^{p-1} (c + p\mathbb{Z}_p), \tag{1}$$

where $c + p\mathbb{Z}_p := \{y \in \mathbb{Q}_p \mid |y - c|_p \leq 1/p\}$. Moreover, by the nonarchimedean property of the p-adic norm, $\mathbb{Z}_p$ is closed under addition and hence is a ring. It is called the ring of *p-adic integers*, and $\mathbb{Z}$ is dense in $\mathbb{Z}_p$. Note that $\mathbb{Z}_p$ is compact and thus complete. (For general references on p-adic analysis, see, e.g., [4, 11].) It is also known that there are topological models of $\mathbb{Z}_p$ in the Euclidean space $\mathbb{R}^d$ as fractal spaces such as the Cantor set and the Sierpiński gasket [11, §I.2.5]. In fact, $\mathbb{Z}_p$ is homeomorphic to the ternary Cantor set. It is thus natural to wonder what exactly is the nonarchimedean (or p-adic) analogue of the ternary Cantor set. We will answer this question in §2.

1.2. Ternary Cantor set

The classical ternary Cantor set, denoted by $\mathcal{C}$, is the set that remains after iteratively removing the open middle third subinterval(s) from the closed unit interval $C_0 = [0, 1]$. The construction is illustrated in Figure 1. Hence, the ternary (or archimedean) Cantor set $\mathcal{C}$ is equal to $\bigcap_{n=0}^{\infty} C_n$.

For comparison with our results in the nonarchimedean case, we state without proof the following well-known results (see, e.g., [1, Ch. 9] and [2, p. 50]):

Theorem 1.1. *The ternary Cantor set $\mathcal{C}$ is self-similar. More specifically, it is the unique nonempty, compact invariant set in $\mathbb{R}$ generated by the family $\{\Phi_1, \Phi_2\}$*

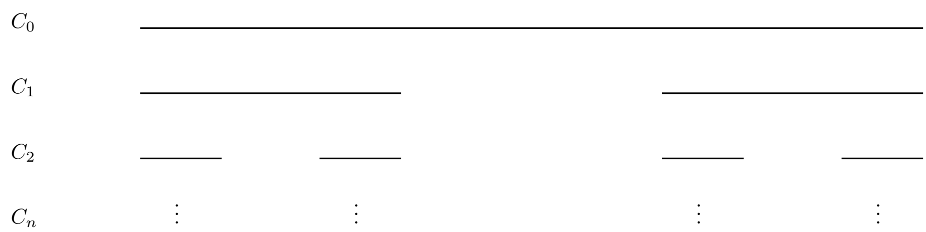

FIGURE 1. Construction of the archimedean Cantor set $\mathcal{C} = \bigcap_{n=0}^{\infty} C_n$.

of similarity contraction mappings of $[0,1]$ *into itself, where* $\Phi_1(x) = x/3$ *and* $\Phi_2(x) = x/3 + 2/3$. *That is,*

$$\mathcal{C} = \Phi_1(\mathcal{C}) \cup \Phi_2(\mathcal{C}).$$

Theorem 1.2. *The Cantor set is characterized by the ternary expansion of its elements as*

$$\mathcal{C} = \left\{ c \in [0,1] \;\middle|\; c = a_0 + \frac{a_1}{3} + \frac{a_2}{3^2} + \cdots,\ a_i \in \{0,2\},\ \forall i \geq 0 \right\}.$$

We note that, as usual, we choose the nonrepeating ternary expansion here. Such a precaution will not be needed in §2 for the elements of $\mathbb{Q}_3$ because the 3-adic expansion is unique.

1.3. Cantor fractal string

The *archimedean* (or *real*) *Cantor string* $\mathcal{CS}$ is defined as the complement of the ternary Cantor set in the closed unit interval $[0,1]$. By construction, the topological boundary of $\mathcal{CS}$ is the ternary Cantor set $\mathcal{C}$. The Cantor string is one of the simplest and most important examples in the research monographs [7, 8] by Lapidus and van Frankenhuijsen. Indeed, it is used throughout those books to illustrate and motivate the general theory; see also, e.g., [5] and [6]. From the point of view of the theory of fractal strings and their complex dimensions [7, 8], it suffices to consider the sequence $\{l_n\}_{n\in\mathbb{N}}$ of lengths associated to $\mathcal{CS}$. More specifically, these are the lengths of the intervals of which the bounded open set $\mathcal{CS} \subset \mathbb{R}$ is composed. Accordingly, the Cantor string consists of $1 = m_1$ interval of length $l_1 = 1/3$, $2 = m_2$ intervals of length $l_2 = 1/9$, $4 = m_3$ intervals of length $l_3 = 1/27$, and so on; see Figure 2.

Important information about the geometry of $\mathcal{CS}$, e.g., the Minkowski dimension and the Minkowski measurability ([5–8]), is contained in its *geometric zeta function*

$$\zeta_{\mathcal{CS}}(s) := \sum_{n=1}^{\infty} m_n \cdot l_n^s = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^{ns}} = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}} \qquad \text{for } \Re(s) > D, \tag{2}$$

where $D = \log 2/\log 3$ is the *Minkowski dimension* of the ternary Cantor set. In addition, $\zeta_{\mathcal{CS}}$ can be extended to a meromorphic function on the entire complex

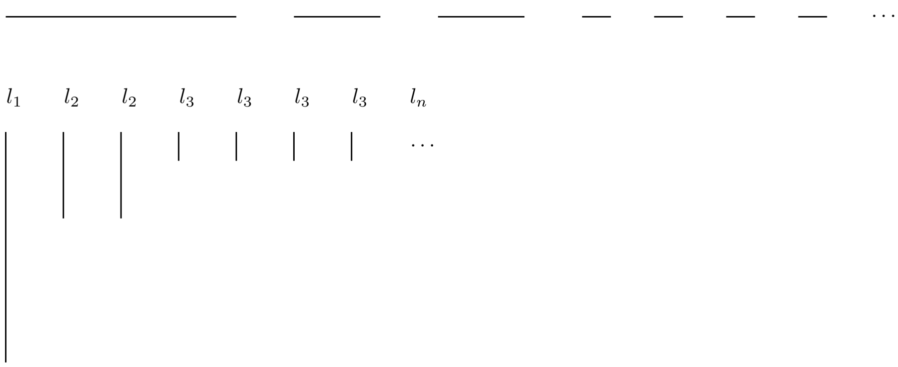

FIGURE 2. Cantor string (above); Cantor string viewed as a fractal harp (below).

plane $\mathbb{C}$, as given by the last expression in (2). The corresponding set of poles of $\zeta_{\mathcal{CS}}$ is then given by

$$\mathcal{D}_{\mathcal{CS}} = \{D + \imath\nu\mathbf{p} \mid \nu \in \mathbb{Z}\}, \tag{3}$$

where $\mathbf{p} = 2\pi/\log 3$ is the *oscillatory period* of $\mathcal{CS}$. Here and henceforth, we let $\imath := \sqrt{-1}$. The set $\mathcal{D}_{\mathcal{CS}}$ is called the set of *complex dimensions* of the Cantor string; see Figure 5 in §3.

The general theme of the monographs [7, 8] is that *the complex dimensions describe oscillations in the geometry and the spectrum of a fractal string.* In particular, there are oscillations of order D in the geometry of $\mathcal{CS}$ and therefore its boundary, the Cantor set, is *not* Minkowski measurable; see [6], [8, §1.1.2].

In §3, we will obtain a nonarchimedean (or p-adic) analogue of the Cantor string and establish its main properties.

2. Nonarchimedean Cantor set

Let $\mathbb{Z}_p$ be the set of p-adic integers. The *p-adic ball* with center $a \in \mathbb{Q}_p$ and radius p^{-n}, $n \in \mathbb{Z}$, is the set

$$a + p^n\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x - a|_p \leq p^{-n}\}.$$

Two interesting "nonarchimedean phenomena" are that each point of the p-adic ball is a center and a p-adic ball is both open and closed. Moreover, every interval[1] in $\mathbb{Q}_p$ can be canonically decomposed into p equally long subintervals, as in (1).

Consider the ring $\mathbb{Z}_3$ of 3-adic integers. In a procedure reminiscent of the construction of the classical Cantor set, we construct the *nonarchimedean Cantor set.* First, we subdivide $T_0 = \mathbb{Z}_3$ into three equally long subintervals. We then remove the "middle" third and call T_1 the remaining set: $T_1 = 0 + 3\mathbb{Z}_3 \cup 2 + 3\mathbb{Z}_3$. We then repeat this process with each of the remaining subintervals, i.e., $0 + 3\mathbb{Z}_3$

[1] We shall sometimes call the ball $a + p^n\mathbb{Z}_p$ an "interval".

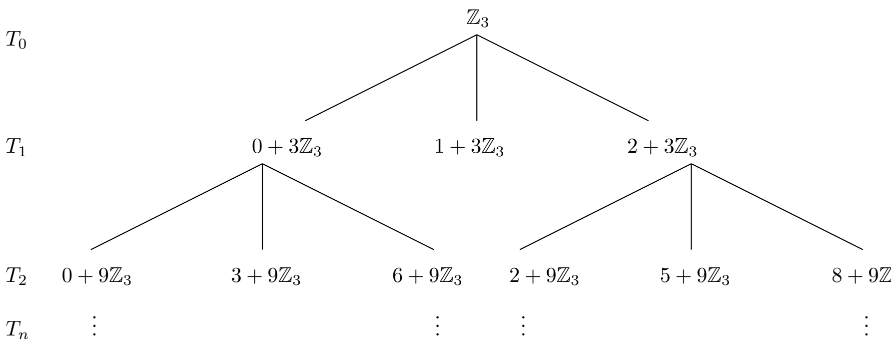

FIGURE 3. Construction of the nonarchimedean Cantor set $\mathcal{C}_3 = \bigcap_{n=0}^{\infty} T_n$.

and $2+3\mathbb{Z}_3$. Finally, we define the nonarchimedean Cantor set $\mathcal{C}_3$ to be $\bigcap_{n=0}^{\infty} T_n$; see Figure 3. The nonarchimedean analogue of Theorem 1.1 is given by Theorem 2.1:

Theorem 2.1. *The nonarchimedean Cantor set $\mathcal{C}_3$ is self-similar. More specifically, it is the unique nonempty, compact invariant set in $\mathbb{Q}_p$ generated by the family $\{\Psi_1, \Psi_2\}$ of similarity contraction mappings of $\mathbb{Z}_3$ into itself, where*

$$\Psi_1(x) = 3x \quad \text{and} \quad \Psi_2(x) = 3x + 2. \tag{4}$$

That is,

$$\mathcal{C}_3 = \Psi_1(\mathcal{C}_3) \cup \Psi_2(\mathcal{C}_3).$$

Proof. From Figure 4, we can see that $\Psi_1(T_n) \cup \Psi_2(T_n) = T_{n+1}$ for all $n \geq 0$. Since each Ψ_i is injective ($i = 1, 2$), we have $\Psi_i(\mathcal{C}_3) = \Psi_i(\bigcap_n T_n) = \bigcap_n \Psi_i(T_n)$. Therefore,

$$\Psi_1(\mathcal{C}_3) \cup \Psi_2(\mathcal{C}_3) = \bigcap_{n=0}^{\infty} (\Psi_1(T_n) \cup \Psi_2(T_n)) = \bigcap_{n=0}^{\infty} T_{n+1} = \mathcal{C}_3.$$

The contraction mapping principle, applied to the complete metric space of all nonempty compact subsets of $\mathbb{Z}_3$,[2] equipped with the Hausdorff metric induced by the 3-adic norm, shows that there is a unique invariant set of the family of similarity contraction mappings $\{\Psi_1, \Psi_2\}$. We refer to Hutchinson's paper [3] for a detailed argument in the case of arbitrary complete metric spaces.

Theorem 2.2. *Let $W_k = \{1, 2\}^k$ be the set of all finite words, on two symbols, of a given length $k \geq 0$. Then*

$$\mathcal{C}_3 = \bigcap_{k=0}^{\infty} \bigcup_{w \in W_k} \Psi_w(\mathbb{Z}_3),$$

where $\Psi_w := \Psi_{w_k} \circ \cdots \circ \Psi_{w_1}$ for $w = (w_1, \ldots, w_k) \in W_k$ and the maps Ψ_{w_i} are as in equation (4).

[2]Recall from §1.1 that $\mathbb{Z}_3$ itself is a complete metric space.

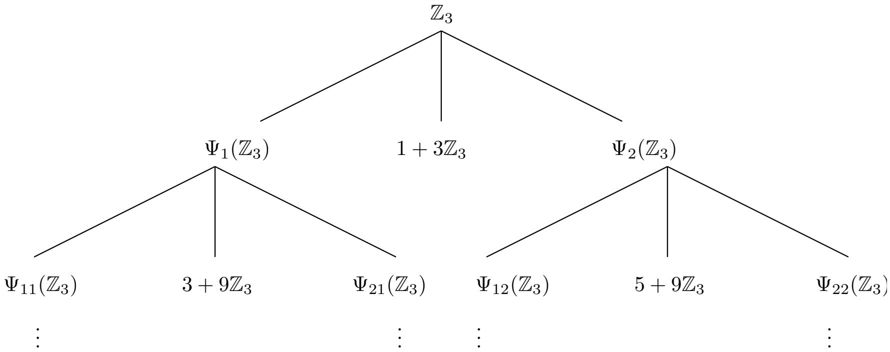

FIGURE 4. Construction of the nonarchimedean Cantor set via an iterated function scheme (IFS).

Proof. For each $k = 0, 1, 2, \ldots$, we have

$$\bigcup_{w \in W_k} \Psi_w(\mathbb{Z}_3) = T_k.$$

Hence, in light of Theorem 2.1, the result follows at once from the definition of $\mathcal{C}_3$; see Figure 4.

The following result is the nonarchimedean analogue of Theorem 1.2:

Theorem 2.3. *The nonarchimedean Cantor set is characterized by the* 3*-adic expansion of its elements. That is,*

$$\mathcal{C}_3 = \left\{\kappa \in \mathbb{Z}_3 \mid \kappa = a_0 + a_1 3 + a_2 3^2 + \cdots,\ a_i \in \{0, 2\},\ \forall i \geq 0\right\}.$$

Proof. First of all, observe that the inverses of Ψ_1 and Ψ_2 are, respectively,

$$\Psi_1^{-1}(x) = \frac{x}{3} \quad \text{and} \quad \Psi_2^{-1}(x) = \frac{x-2}{3}.$$

Secondly, it is clear that for $a_i' \in \{0, 1, 2\}$ and $i = 0, 1, \ldots$,

$$a_0' + a_1' 3 + a_2' 3^2 + \cdots \in 1 + 3\mathbb{Z}_3 \ \Leftrightarrow\ a_0' = 1. \tag{5}$$

Now, let $\kappa = a_0 + a_1 3 + a_2 3^2 + \cdots \in \mathbb{Z}_3$ and suppose that some coefficients in its 3-adic expansion are 1's. We will show that κ must then be in the image of $1+3\mathbb{Z}_3$ under some composition of the maps Ψ_1 and Ψ_2. Let $l \in \mathbb{N}$ be the first index such that $a_l = 1$. Hence, $a_j = 0$ or 2 for all $j < l$. If $a_0 = 0$, then we apply Ψ_1^{-1} to κ, and if $a_0 = 2$, then we apply Ψ_2^{-1} to κ. In both cases, we have

$$\Psi_i^{-1}(\kappa) = a_1 + a_2 3 + \cdots + a_l 3^{l-1} + a_{l+1} 3^l + \cdots. \tag{6}$$

Depending on whether $a_1 = 0$ or 2, we apply Ψ_1^{-1} or Ψ_2^{-1}, respectively, to the above 3-adic expansion (6). Proceeding in this manner, we will get

$$\Psi_w^{-1}(\kappa) = a_l + a_{l+1} 3 + \cdots,$$

which is in $1+3\mathbb{Z}_3$ for some $k \in \mathbb{N}$ and $w \in W_k$. Thus $\kappa \in \Psi_w(1+3\mathbb{Z}_3)$. Since $(1+3\mathbb{Z}_3) \cap \mathcal{C}_3 = \emptyset$ and Ψ_w is injective, we deduce that $\kappa \notin \mathcal{C}_3$. Therefore, all of the digits of $\kappa \in \mathcal{C}_3$ must lie in $\{0, 2\}$.

Conversely, suppose that all of the coefficients in $\kappa = a_0 + a_1 3 + a_2 3^2 + \cdots \in \mathbb{Z}_3$ are 0's or 2's. Then, by the above observation (5), $\kappa \notin 1+3\mathbb{Z}_3$. Moreover, $\kappa \notin \Phi_w(1+3\mathbb{Z}_3)$ for any $w \in W_k$, $k = 0, 1, 2, \ldots$, since none of the coefficients a_i is equal to 1. That is,

$$\kappa \notin \bigcup_{k=0}^{\infty} \bigcup_{w \in W_k} \Phi_w(1+3\mathbb{Z}_3) =: B.$$

But $\mathcal{C}_3 \cap B = \emptyset$ and $\mathcal{C}_3 \cup B = \mathbb{Z}_3$, as can be seen from equation (8) and Theorem 3.3. Hence, $\kappa \in \mathcal{C}_3$, as desired.

Theorem 2.4. *The ternary Cantor set $\mathcal{C}$ and the nonarchimedean Cantor set $\mathcal{C}_3$ are homeomorphic.*

Proof. Let $\phi : \mathcal{C} \to \mathcal{C}_3$ be the map sending

$$\sum_{i=0}^{\infty} a_i 3^{-i} \mapsto \sum_{i=0}^{\infty} a_i 3^i, \tag{7}$$

where $a_i \in \{0, 2\}$ for all $i \geq 0$. We note that on the left-hand side of (7), we use the ternary expansion in $\mathbb{R}$, whereas on the right-hand side we use the 3-adic expansion in $\mathbb{Q}_3$. Then, clearly, ϕ is a continuous bijective map from $\mathcal{C}$ to $\mathcal{C}_3$. Since both $\mathcal{C}$ and $\mathcal{C}_3$ are compact spaces in their respective natural metric topologies, ϕ is a homeomorphism.

Remark 2.5. In view of Theorem 2.4, like its archimedean counterpart $\mathcal{C}$, the nonarchimedean Cantor set $\mathcal{C}_3$ is totally disconnected, uncountably infinite and has no isolated points.

3. Nonarchimedean Cantor string

The *nonarchimedean* (or *p-adic*) *Cantor string* is defined to be

$$\mathcal{CS}_3 := (1+3\mathbb{Z}_3) \cup (3+9\mathbb{Z}_3) \cup (5+9\mathbb{Z}_3) \cup \cdots = \mathbb{Z}_3 \setminus \mathcal{C}_3, \tag{8}$$

the complement of $\mathcal{C}_3$ in $\mathbb{Z}_3$; see the "middle" parts of Figure 3. Therefore, by analogy with the relationship between the archimedean Cantor set and Cantor string, the nonarchimedean Cantor set $\mathcal{C}_3$ can be thought of as some kind of "boundary" of the nonarchimedean Cantor string. Certainly, $\mathcal{C}_3$ is not the topological boundary of $\mathcal{CS}_3$ because the latter boundary is empty.

Since $\mathbb{Q}_p$ is a locally compact group, there is a unique translation invariant Haar measure μ_H, normalized so that $\mu_H(\mathbb{Z}_p) = 1$, and hence $\mu_H(a+3^n\mathbb{Z}_3) = 3^{-n}$; see [4], [11]. As in the real case in §1.3, we may identify $\mathcal{CS}_3$ with the sequence of lengths $l_n = 3^{-n}$ with multiplicities $m_n = 2^{n-1}$ for $n \in \mathbb{N}$.

The following theorem provides the exact analogue of equations (2) and (3):

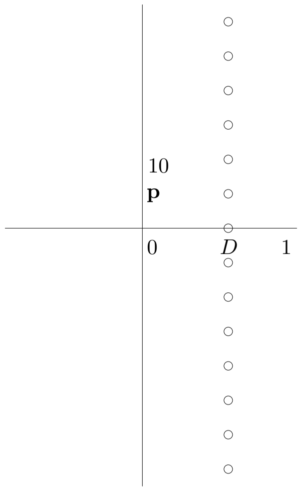

FIGURE 5. The set of complex dimensions, $\mathcal{D}_{\mathcal{CS}} = \mathcal{D}_{\mathcal{CS}_3}$, of the archimedean and nonarchimedean Cantor strings, $\mathcal{CS}$ and $\mathcal{CS}_3$.

Theorem 3.1. *The geometric zeta function of the nonarchimedean Cantor string is meromorphic in all of* $\mathbb{C}$ *and is given by*

$$\zeta_{\mathcal{CS}_3}(s) = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}}. \tag{9}$$

Hence, the set of complex dimensions of $\mathcal{CS}_3$ *is given by*

$$\mathcal{D}_{\mathcal{CS}_3} = \{D + \imath\nu\mathbf{p} \mid \nu \in \mathbb{Z}\}, \tag{10}$$

where $D = \log 2/\log 3$ *is the dimension of* $\mathcal{CS}_3$ *and* $\mathbf{p} = 2\pi/\log 3$ *is its oscillatory period.*

Proof. By definition (see [9, 10]), the geometric zeta function of $\mathcal{CS}_3$ is given by

$$\begin{aligned}\zeta_{\mathcal{CS}_3}(s) &:= (\mu_H(1 + 3\mathbb{Z}_3))^s + (\mu_H(3 + 9\mathbb{Z}))^s + (\mu_H(5 + 9\mathbb{Z}_3))^s + \cdots \\ &= \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^{ns}} = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}} \qquad \text{for } \Re(s) > \log 2/\log 3.\end{aligned}$$

Furthermore, the meromorphic extension of $\zeta_{\mathcal{CS}_3}$ to all of $\mathbb{C}$ is given by the last expression in the above equation. The *complex dimensions* of $\mathcal{CS}_3$, defined as the poles of $\zeta_{\mathcal{CS}_3}$, are all the solutions ω of the equation $1 - 2 \cdot 3^{-\omega} = 0$. These are precisely of the form

$$\omega = \frac{\log 2}{\log 3} + \imath\nu\frac{2\pi}{\log 3}, \qquad \nu \in \mathbb{Z}.$$

Remark 3.2. In [9, 10], we prove that D is the Minkowski dimension of $\mathcal{CS}_3 \subset \mathbb{Z}_3$. Clearly, D is also the abscissa of convergence of the Dirichlet series defining $\zeta_{\mathcal{CS}_3}$.

The following result was used in the second part of the proof of Theorem 2.3:

Theorem 3.3. *With the same notation as in Theorem 2.2, we have*

$$\mathcal{CS}_3 = \bigcup_{k=0}^{\infty} \bigcup_{w \in W_k} \Psi_w(1 + 3\mathbb{Z}_3).$$

Proof. For each $k = 0, 1, 2, \ldots$, we let $\widetilde{T}_{k+1} = \mathbb{Z}_3 \backslash T_{k+1}$, the complement of T_{k+1} in $\mathbb{Z}_3$. Then

$$\widetilde{T}_{k+1} = \bigcup_{w \in W_k} \Psi_w(1 + 3\mathbb{Z}_3).$$

Hence, in light of Theorem 2.1,

$$\bigcup_{k=0}^{\infty} \bigcup_{w \in W_k} \Psi_w(1 + 3\mathbb{Z}_3) = \bigcup_{k=0}^{\infty} \widetilde{T}_{k+1} = \Big(\bigcap T_{k+1}\Big)^{\sim} = \widetilde{\mathcal{C}}_3 = \mathcal{CS}_3,$$

by the definitions of $\mathcal{C}_3$ and $\mathcal{CS}_3$. See Figure 6.

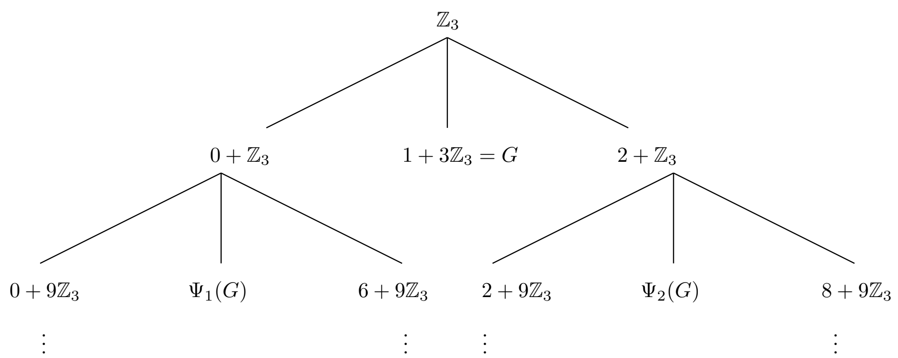

Figure 6. Construction of the nonarchimedean Cantor string via an IFS.

Remark 3.4. The above theorem shows that $G = 1 + 3\mathbb{Z}_3$ is the *generator* of the nonarchimedean Cantor string. This is a particular case of a more general construction of self-similar p-adic fractal strings [9, 10]. Moreover, $\mathcal{CS}_3$ is *not* Minkowski measurable as a subset of $\mathbb{Q}_3$. In fact, in contrast to the archimedean case ([8, Theorems 8.23 and 8.36]), self-similar p-adic fractal strings are *never* Minkowski measurable.

Acknowledgments

We would like to thank Scot Childress, Erin Pearse and Steffen Winter for helpful discussions and comments on a preliminary version of this paper.

The research of the first author was partially supported by the US National Science Foundation under the grants DMS-0070497 and DMS-0707524.

References

[1] K. J. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*. Wiley, Chichester, 1990.

[2] V. P. Gupta and P. K. Jain, *Lebesgue Measure and Integration*. Wiley, New Delhi, 1986.

[3] J. E. Hutchinson, *Fractals and self-similarity*. Indiana Univ. Math. J. **30** (1981), 713–747.

[4] N. Koblitz, *p-adic Numbers, p-adic Analysis, and Zeta-Functions*. 2nd ed., Springer, New York, 1984.

[5] M. L. Lapidus, *Spectral and fractal geometry: From the Weyl–Berry conjecture for the vibrations of fractal drums to the Riemann zeta-function*. In: Differential Equations and Mathematical Physics (Birmingham, 1990), C. Bennewitz (ed.), Academic Press, New York, 1992, 151–182.

[6] M. L. Lapidus and C. Pomerance, *The Riemann zeta-function and the one-dimensional Weyl–Berry conjecture for fractal drums*. Proc. London Math. Soc. (3) **60** (1993), 41–69.

[7] M. L. Lapidus and M. van Frankenhuijsen, *Fractal Geometry and Number Theory: Complex dimensions of fractal strings and zeros of zeta functions*. Birkhäuser, Boston, 2000.

[8] M. L. Lapidus and M. van Frankenhuijsen, *Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and spectra of fractal strings*. Springer, New York, 2006.

[9] M. L. Lapidus and H. Lũ', *Self-similar p-adic fractal strings and their complex dimensions*. In preparation.

[10] H. Lũ', *p-adic fractal strings and their complex dimensions*, Ph.D. Dissertation, Univ. California, Riverside, 2007.

[11] A. M. Robert, *A Course in p-adic Analysis*. Springer, New York, 2000.

[12] V. S. Vladimirov, I. V. Volovich, and E. I. Zelenov, *p-Adic Analysis and Mathematical Physics*. World Sci., Singapore, 1994.

Michel L. Lapidus
Department of Mathematics
University of California
Riverside, CA 92521, USA
e-mail: `lapidus@math.ucr.edu`

Hùng Lũ'
Department of Mathematics
Hawai'i Pacific University
Honolulu, HI 96813, USA
e-mail: `hlu@hpu.edu`