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Dirac operators and spectral triples for some fractal sets built on curves

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Abstract

We construct spectral triples and, in particular, Dirac operators, for the algebra of continuous functions on certain compact metric spaces. The triples are countable sums of triples where each summand is based on a curve in the space. Several fractals, like a finitely summable infinite tree and the Sierpinski gasket, fit naturally within our framework. In these cases, we show that our spectral triples do describe the geodesic distance and the Minkowski dimension as well as, more generally, the complex fractal dimensions of the space. Furthermore, in the case of the Sierpinski gasket, the associated Dixmier-type trace coincides with the normalized Hausdorff measure of dimension $\log 3 / \log 2$.
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0. Introduction

Consider a smooth compact spin Riemannian manifold M and the Dirac operator ∂_M associated with a fixed Riemannian connexion over the spinor bundle S. Let D denote the extension of ∂_M to H, the Hilbert space of square-integrable sections (or spinors) of S. Then Alain Connes showed that the geodesic distance, the dimension of M and the Riemannian volume measure can all be described via the operator D and its associated *Dixmier trace*. This observation makes it possible to reformulate the geometry of a manifold in terms of a representation of the algebra of continuous functions as operators on a Hilbert space on which an unbounded self-adjoint operator D acts. The geometric structures are thus described in such a way that points in the manifold are not mentioned at all. It is then possible to replace the commutative algebra of coordinates by a not necessarily commutative algebra, and a way of expressing geometric properties for noncommutative algebras has been opened.

Connes, alone or in collaborations, has extensively developed the foregoing idea. (See, for example, the paper [5], the book [6] and the recent survey article [8], along with the relevant references therein.) He has shown how an unbounded Fredholm module over the algebra $\mathcal{A}$ of coordinates of a possibly noncommutative space X specifies some elements of the (quantized) differential and integral calculus on X as well as the metric structure of X. An *unbounded Fredholm module* $(\mathcal{H}, D)$ consists of a representation of $\mathcal{A}$ as bounded operators on a Hilbert space $\mathcal{H}$ and an unbounded self-adjoint operator D on $\mathcal{H}$ satisfying certain axioms. This operator is then called a *Dirac operator* and, if the representation is faithful, the triple $(\mathcal{A}, \mathcal{H}, D)$ is called a *spectral triple*.

The introduction of the concept of a spectral triple for a not necessarily commutative algebra $\mathcal{A}$ makes it possible to assign a subalgebra of $\mathcal{A}$ as playing the role of the algebra of the infinitely differentiable functions, even though expressions of the type $(f(x) - f(y))/(x - y)$ do not make sense anymore.

With this space-free description of some elements of differential geometry at our disposal, a new way of investigating the geometry of fractal sets seems to be possible. Fractal sets are nonsmooth in the traditional sense, so at first it seems quite unreasonable to try to apply the methods of differential geometry in the study of such sets. On the other hand, any compact set $\mathcal{T}$ is completely described as the spectrum of the C*-algebra $C(\mathcal{T})$ consisting of the continuous complex functions on it. This means that if we can apply Connes' space-free techniques to a dense subalgebra of $C(\mathcal{T})$, then we do get some insight into the sort of geometric structures on the compact space $\mathcal{T}$ that are compatible with the given topology.

Already, in their unpublished work [9], Alain Connes and Dennis Sullivan have developed a 'quantized calculus' on the limit sets of 'quasi-Fuchsian' groups, including certain Julia sets. Their results are presented in [6, Chapter 4, Sections 3.γ and 3.ϵ], along with related results of Connes on Cantor-type sets (see also [7]). The latter are motivated in part by the work of Michel Lapidus and Carl Pomerance [32] or Helmut Maier [31] on the geometry and spectra of fractal strings and their associated zeta functions (now much further expanded in the theory of 'complex fractal dimensions' developed in the books [29,30]).

In particular, in certain cases, the Minkowski dimension and the Hausdorff measure can be recovered from operator algebraic data. Work in this direction was pursued by Daniele Guido and Tommaso Isola in several papers, [16–18].

Earlier, in [27], using the results and methods of [21] and [6] (including the notion of 'Dixmier trace'), Lapidus has constructed an analogue of (Riemannian) volume measures on finitely ramified or p.c.f. self-similar fractals and related them to the notion of 'spectral dimension,' which

gives the asymptotic growth rate of the spectra of Laplacians on these fractals. (See also [22] where these volume measures were later precisely identified, for a class of fractals including the Sierpinski gasket.) In his programmatic paper [28], building upon [27], he then investigated in many different ways the possibility of developing a kind of *noncommutative fractal geometry*, which would merge aspects of *analysis on fractals* (as now presented e.g. in [20]) and *Connes' noncommutative geometry* [6]. (See also parts of [26] and [27].) Central to [28] was the proposal to construct suitable spectral triples that would capture the geometric and spectral aspects of a given self-similar fractal, including its metric structure. Much remains to be done in this direction, however, but the present work, along with some of its predecessors mentioned above and below, address these issues from a purely geometric point of view. In later work, we hope to be able to extend our construction to address some of the more spectral aspects.

In [2], Erik Christensen and Cristina Ivan constructed a spectral triple for the approximately finite-dimensional (AF) C*-algebras. The continuous functions on the Cantor set form an AF C*-algebra since the Cantor set is totally disconnected. Hence, it was quite natural to try to apply the general results of that paper for AF C*-algebras to this well-known example. In this manner, they showed in [2] once again how suitable noncommutative geometry may be to the study of the geometry of a fractal. Since then, the authors of the present article have searched for possible spectral triples associated to other known fractals. The hope is that these triples may be relevant to both *fractal geometry* and *analysis on fractals*. We have been especially interested in the Sierpinski gasket, a well-known nowhere differentiable planar curve, because of its key role in the development of harmonic analysis on fractals. (See, for example, [1,20,39,40].) We present here a spectral triple for this fractal which recovers its geometry very well. In particular, it captures some of the visually detectable aspects of the gasket.

We will now give a more detailed description of the Sierpinski gasket and of the spectral triple we construct in that case. The way of looking at the gasket, upon which our construction is based, is as the closure of the limit of an increasing sequence of graphs. The pictures in Fig. 1 illustrate this. We have chosen the starting approximation, $\mathcal{SG}_0$, as a triangle, and as a graph it consists of 3 vertices and 3 edges. The construction is of course independent of the scale, but we will assume that we work in the Euclidean plane and that the usual concept of length is given. We then fix the size of the largest triangle such that all the sides have length $2\pi/3$. Then, for any natural number n, the nth approximation $\mathcal{SG}_n$ is constructed from $\mathcal{SG}_{n-1}$ by adding 3^n new vertices and 3^n new edges of length $2^{1-n}\pi/3$. It is possible to construct a spectral triple for the gasket by constructing a spectral triple for each new edge introduced in one of the graphs $\mathcal{SG}_n$, but in order to let the spectral triples describe the holes too, and not only the lines, we have chosen to look at this iterative construction as an increasing union of triangles. Hence, $\mathcal{SG}_0$ consists of one equilateral triangle with circumference (i.e., perimeter) equal to 2π and $\mathcal{SG}_1$ consists of the union of 3 equilateral triangles each of circumference π. Continuing in this manner, we get for any natural number n that $\mathcal{SG}_n$ is a union of 3^n equilateral triangles of side length $2^{1-n}\pi/3$. The advantage of this description is that, for any equilateral triangle of circumference $2^{1-n}\pi$, there is
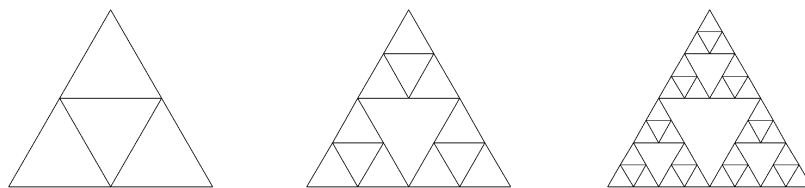

Fig. 1. The first 3 pre-fractal approximations to the Sierpinski gasket.

an obvious way to construct a spectral triple for the continuous functions on such a set. The idea is to view such a triangle as a modified circle of circumference $2^{1-n}\pi$ and then use the standard spectral triple for the continuous functions on such a circle.

The direct sum of all these spectral triples is not automatically a spectral triple, but a minor translation of each of the involved Dirac operators can make a direct sum possible so that we can obtain a spectral triple for the continuous functions on the Sierpinski gasket in such a way that the following geometric structures are described by the triple:

 (i) The metric induced on the Sierpinski gasket by the spectral triple is exactly the *geodesic distance* on the gasket.
 (ii) The spectral triple is *s*-summable for any *s* greater than $\log 3/\log 2$ and not *s*-summable for any *s* smaller than or equal to this number. Hence it has *metric dimension* $\log 3/\log 2$, equal to the *Minkowski dimension* of the gasket.
(iii) The set of (geometric) complex fractal dimensions of the Sierpinski gasket (defined here as the poles of the zeta function of the spectral triple) is given by

$$\left\{\frac{\log 3}{\log 2} + \sqrt{-1} \cdot k \cdot \frac{2\pi}{\log 2} \,\Big|\, k \in \mathbb{Z}\right\} \cup \{1\},$$

much as was suggested in [28].
(iv) For each natural number *n*, let the set of vertices in the graph $\mathcal{SG}_n$ be denoted $\mathcal{V}_n$ and its cardinality $|\mathcal{V}_n|$. Then there exists a state ψ on $C(\mathcal{SG})$ such that for any continuous complex-valued function f on $\mathcal{SG}$,

$$\lim_{n \to \infty} \frac{1}{|\mathcal{V}_n|} \sum_{v \in \mathcal{V}_n} f(v) = \psi(f).$$

This function ψ is a multiple of the positive functional τ on $C(\mathcal{SG})$ which the spectral triple and a Dixmier trace create. Further, we show that ψ coincides with the *normalized Hausdorff measure* on the Sierpinski gasket.

The methods which we have developed for the Sierpinski gasket are based on harmonic analysis on the circle. After having provided some necessary background in Section 1, we will recall in Section 2 the construction of the standard spectral triple for the continuous functions on a circle. Since a continuous simple image of a circle in a compact topological space, say X, induces a *-homomorphism of the continuous functions on this space onto the continuous functions on the circle, such a closed curve in a compact space will induce an unbounded Fredholm module over the algebra of continuous functions on X.

The results for continuous images of circles in the space X can be extended to simple, i.e. not self-intersecting, continuous images of intervals. We call such an image a simple continuous curve in X. The method is based on the standard embedding of an interval in a circle, by forming a circle from an interval as the union of the interval and a copy of the same interval and then gluing the endpoints. This is done formally in Sections 3 and 4. We can then construct spectral triples for $C(X)$ which are formed as direct sums of unbounded Fredholm modules associated to some curves in the space. This construction of spectral triples built on curves (and loops) is given in Section 5 and makes good sense and is further refined for finite graphs (Section 6), as well as for suitable infinite graphs such as trees (Section 7). In the latter section, we illustrate

our results by considering the fractal tree (Cayley graph) associated to $\mathbb{F}_2$, the noncommutative free group of two generators. In Section 8, we then apply our results and study in detail their consequences for the important example of the Sierpinski gasket, as described above. Finally, in Section 9, we close this paper by discussing several open problems and proposing directions for further research.

1. Notation and preliminaries

In the book [6], Connes explains how a lot of geometric properties of a compact space X can be described via the algebra of continuous functions C(X), representations of this algebra and some unbounded operators on a Hilbert space like differential operators. The concepts mentioned do not rely on the commutativity of the algebra C(X), so it is possible to consider a general non-commutative C*-algebra, its representations and some unbounded operators on a Hilbert space as noncommutative versions of structures which in the first place have been expressed in terms of sets and geometric properties. In particular, the closely related notions of unbounded Fredholm module and spectral triple are fundamental and we will recall them here.

Definition 1.1. Let $\mathcal{A}$ be a unital C*-algebra. An *unbounded Fredholm module* (H, D) over $\mathcal{A}$ consists of a Hilbert space H which carries a unital representation π of $\mathcal{A}$ and an unbounded, self-adjoint operator D on H such that

 (i) the set $\{a \in \mathcal{A} \mid [D, \pi(a)]$ is densely defined and extends to a bounded operator on $H\}$ is a dense subset of $\mathcal{A}$,
(ii) the operator $(I + D^2)^{-1}$ is compact.

If, in addition, $\text{tr}((I + D^2)^{-p/2}) < \infty$ for some positive real number p, then the unbounded Fredholm module, is said to be *p-summable*, or just *finitely summable*. The number $\mathfrak{d}_{ST}$ given by

$$\mathfrak{d}_{ST} := \inf\big\{ p > 0 \,\big|\, \text{tr}\big((I + D^2)^{-p/2}\big) < \infty \big\}$$

is called the *metric dimension* of the unbounded Fredholm module.

Definition 1.2. Let $\mathcal{A}$ be a unital C*-algebra and (H, D) an unbounded Fredholm module over $\mathcal{A}$. If the underlying representation π is faithful, then $(\mathcal{A}, H, D)$ is called a *spectral triple*.

For notational simplicity, we will denote by $(\mathcal{A}, H, D)$ either a spectral triple or an unbounded Fredholm module, whether or not the underlying representation is faithful.

We will refer to Connes' book [6] on several occasions later in this article. As a standard reference on the theory of operator algebras, we use the books by Kadison and Ringrose [19].

The emphasis of this article is actually not very much on noncommutative C*-algebras and noncommutative geometry, but our interest here is to see to what extend the spectral triples (unbounded Fredholm modules) of noncommutative geometry can be used to describe compact spaces of a fractal nature. In the papers [2,3], one can find investigations of this sort for the Cantor set and for general compact metric spaces, respectively. The metrics of compact spaces introduced via noncommutative geometric methods were extensively investigated by Marc Rieffel. (See, for example, the papers [34–37].)

Our standard references for concepts from fractal geometry are Edgar's books [12,13], Falconer's book [15] and Lapidus and Frankenhuijsen's books [29,30].

2. The spectral triple for a circle

Much of the material in this section is well known (see e.g. [14]) but is not traditionally presented in the language of spectral triples. It is useful, however, to discuss it here because to our knowledge, it is only available in scattered references and not in the form we need it. In particular, we will study in some detail the domains of definition of the relevant unbounded operators and derivations.

Let C_r denote the circle in the complex plane with radius $r > 0$ and centered at 0. As usual in noncommutative geometry, we are not studying the circle directly but rather a subalgebra of the algebra of continuous functions on it. From this point of view, it seems easier to look at the algebra of complex continuous $2\pi r$-periodic functions on the real line. We will let $\mathcal{A}C_r$ denote this algebra. We will let $(1/2\pi r)\mathfrak{m}$ denote the normalized Lebesgue measure on the interval $[-\pi r, \pi r]$ and let π_r be the standard representation of $\mathcal{A}C_r$ as multiplication operators on the Hilbert space H_r which is defined by $H_r := L^2([-\pi r, \pi r], (1/2\pi r)\mathfrak{m})$. The space H_r has a canonical orthonormal basis, denoted $(\phi_k^r)_{k\in\mathbb{Z}}$, which consists of functions in $\mathcal{A}C_r$ given by

$$\forall k \in \mathbb{Z}, \quad \phi_k^r(x) := \exp\left(\frac{ikx}{r}\right).$$

These functions are eigenfunctions of the differential operator $\frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}x}$ and the corresponding eigenvalues are $\{k/r \mid k \in \mathbb{Z}\}$. The natural choice for the Dirac operator for this situation is the closure of the restriction of the above operator to the linear span of the basis $\{\phi_k^r \mid k \in \mathbb{Z}\}$. We will let D_r denote this operator on H_r. It is well known that D_r is self-adjoint and that $\mathrm{dom}(D_r)$, the domain of definition of D_r, is given by

$$\forall f \in H_r: \quad f \in \mathrm{dom}\, D_r \quad \Leftrightarrow \quad \sum_{k\in\mathbb{Z}} \frac{k^2}{r^2}\big|\langle f \mid \phi_k^r \rangle\big|^2 < \infty,$$

where $\langle \cdot \mid \cdot \rangle$ is the inner product of H_r.

For an element $f \in \mathrm{dom}\, D_r$, we have $D_r f = \sum_{k\in\mathbb{Z}}(k/r)\langle f \mid \phi_k^r \rangle \phi_k^r$. The self-adjoint operator D_r has spectrum $\{k/r \mid k \in \mathbb{Z}\}$ and each of its eigenvalues has multiplicity 1. Furthermore, any continuously differentiable $2\pi r$-periodic function f on $\mathbb{R}$ satisfies

$$\big[D_r, \pi_r(f)\big] = \pi_r\big(-if'\big),$$

so we obtain a spectral triple associated to the circle C_r in the following manner.

Definition 2.1. The *natural spectral triple*, $ST_n(C_r)$, for the circle algebra $\mathcal{A}C_r$ is defined by $ST_n(C_r) := (\mathcal{A}C_r, H_r, D_r)$.

One of the main ingredients in the arguments to come is the possibility to construct *interesting* spectral triples as direct sums of unbounded Fredholm modules, each of which only carries very little information on the total space. In the case of the natural triples for circles, the number 0 is

always an eigenvalue and hence, if the sum operation is done a countable number of times, the eigenvalue 0 will be of infinite multiplicity for the Dirac operator which is obtained via the direct sum construction. In order to avoid this problem, we will, for the C_r-case, replace the Dirac operator D_r by a slightly modified one, D_r^t, which is the translate of D_r given by

$$D_r^t := D_r + \frac{1}{2r}I.$$

The set of eigenvalues now becomes $\{(2k+1)/2r \mid k \in \mathbb{Z}\}$, but the domain of definition is the same as for D_r and, *in particular*, for any function $f \in \mathcal{A}C_r$, we have $[D_r^t, \pi_r(f)] = [D_r, \pi_r(f)]$. Hence, the translation does not really change the effect of the spectral triple.

Definition 2.2. The *translated spectral triple*, $ST_t(C_r)$, for the circle algebra $\mathcal{A}C_r$ is defined by $ST_t(C_r) := (\mathcal{A}C_r, H_r, D_r^t)$.

The next question is to determine for which functions f from $\mathcal{A}C_r$ the commutator $[D_r^t, \pi_r(f)]$ is bounded and densely defined. This is done in the following lemma which is standard, but which we include since its specific statement cannot be easily found in the way we need it. On the other hand, the proof is based on elementary analysis and is therefore omitted.

Lemma 2.3. *Let $f \in \mathcal{A}C_r$. Then the following conditions are equivalent*:

  (i) *$[D_r^t, \pi_r(f)]$ is densely defined and bounded.*
  (ii) *$f \in \mathrm{dom}(D_r)$ and $D_r f$ is essentially bounded.*
  (iii) *There exists a measurable, essentially bounded function g on the interval $[-\pi r, \pi r]$ such that*

$$\int_{-\pi r}^{\pi r} g(t)\,dt = 0 \quad and \quad \forall x \in [-\pi r, \pi r]: \quad f(x) = f(0) + \int_0^x g(t)\,dt.$$

If the conditions above are satisfied, then $g(x) = (i D_r f)(x)$ a.e.

We will end this section by mentioning some properties of this spectral triple. We will not prove any of the claims below, as they are easy to verify. First, we remark that all the statements below hold for both of the triples $ST_t(C_r)$ and $ST_n(C_r)$, although they are only formulated for $ST_n(C_r)$.

Theorem 2.4. *Let $r > 0$ and let $(\mathcal{A}_r C, H_r, D_r)$ be the $ST_n(C_r)$ circle spectral triple. Then the following two results hold*:

  (i) *The metric, say d_r, induced by the spectral triple $ST_n(C_r)$ on the circle is the geodesic distance on C_r.*
  (ii) *The spectral triple $ST_n(C_r)$ is summable for any $s > 1$, but not for $s = 1$. Hence, it has metric dimension 1.*

3. The interval triple

The standard way to study an interval by means of the theory for the circle is to take two copies of the interval and then glue them together at the endpoints. When working with algebras instead of spaces, this construction is done via an injective homomorphism Φ_α of the continuous functions on the interval $[0, \alpha]$ into the continuous functions on the double interval $[-\alpha, \alpha]$ defined by

$$\forall f \in C([0, \alpha]), \ \forall t \in [-\alpha, \alpha]: \quad \Phi_\alpha(f)(t) := f(|t|).$$

The continuous functions on the interval $[0, \alpha]$ are mapped by this procedure onto the even continuous 2α-periodic continuous functions on the real line, and the theory describing the properties of the spectral triple $(AC_{\alpha/\pi}, H_{\alpha/\pi}, D^t_{\alpha/\pi})$ may be used to describe a spectral triple for the algebra $C([0, \alpha])$. We will start by defining the spectral triple ST_α which we associate to the interval $[0, \alpha]$ before we actually prove that this is a spectral triple. The arguments showing that ST_α is a spectral triple follow from the analogous results for the circle.

Definition 3.1. Given $\alpha > 0$, the α-*interval spectral triple* $ST_\alpha := (\mathcal{A}_\alpha, H_\alpha, D_\alpha)$ is defined by

   (i) $\mathcal{A}_\alpha = C([0, \alpha])$;
  (ii) $H_\alpha = L^2([-\alpha, \alpha], \mathfrak{m}/2\alpha)$, where the measure $\mathfrak{m}/2\alpha$ is the normalized Lebesgue measure;
 (iii) the representation $\pi_\alpha : \mathcal{A}_\alpha \to B(H_\alpha)$ is defined for f in $\mathcal{A}_\alpha$ as the multiplication operator which multiplies by the function $\Phi_\alpha(f)$;
 (iv) an orthonormal basis $\{e_k \mid k \in \mathbb{Z}\}$ for H_α is given by $e_k(x) := \exp(i\pi kx/\alpha)$ and D_α is the self-adjoint operator on H_α which has all the vectors e_k as eigenvectors and such that $D_\alpha e_k = (\pi k/\alpha)e_k$ for each $k \in \mathbb{Z}$.

Let us look at the expression $[D_\alpha, \pi_\alpha(f)]$, for a smooth function f. Then it is well known that $[D_\alpha, \pi_\alpha(f)] = \pi_\alpha(D_\alpha f)$, and we want to remark that for an even function f we have that $D_\alpha f$ is an odd function. Hence, here in the most standard commutative example, we already meet a noncommutative aspect of the classical theory, in the sense that for any function f in $\mathcal{A}_\alpha$ the commutator $[D_\alpha, \pi_\alpha(f)]$, if it exists, is no longer in the image of $\pi_\alpha(\mathcal{A}_\alpha)$.

The following proposition demonstrates that many even continuous functions f do have bounded commutators with D_α. The result follows directly from Lemma 2.3.

Proposition 3.2. *Let f be a continuous real and even function on the interval $[-\alpha, \alpha]$ such that f is boundedly and continuously differentiable outside a set of finitely many points. Then f is in the domain of definition of D_α and $D_\alpha f$ is bounded outside a set of finitely many points.*

The close connection between the spectral triples for the circle and the interval yields immediately the following corollary of Theorem 2.4.

Theorem 3.3. *Given $\alpha > 0$, let $(\mathcal{A}_\alpha, H_\alpha, D_\alpha)$ be the α-interval spectral triple. Then, for any pair of reals s, t such that $0 \leqslant s < t \leqslant \alpha$, we have*

$$|t - s| = \sup\{|f(t) - f(s)| \mid \|[D_\alpha, \pi_\alpha(f)]\| \leqslant 1\}.$$

Furthermore, the triple is summable for any real $s > 1$ and not summable for $s = 1$. Hence, it has metric dimension 1.

4. The r-triple, ST_r

Let now T be a compact and Hausdorff space and $r : [0, \alpha] \to T$ a continuous and injective mapping. The image in T, i.e. the set of points $\mathcal{R} = r([0, \alpha])$, is called a continuous curve and r is then a parameterization of $\mathcal{R}$. As usual, a continuous curve may have many parameterizations and one of the possible uses of the concept of a spectral triple is that it can help us to distinguish between various parameterizations of the same continuous curve. Below we will associate to a parameterization r of a continuous curve $\mathcal{R}$ an unbounded Fredholm module, ST_r. After having read the statement of the following proposition, it will be clear how this module may be defined.

Proposition 4.1. *Let $r : [0, \alpha] \to T$ be a continuous injective mapping and $(\mathcal{A}_\alpha, H_\alpha, D_\alpha)$ the α-interval spectral triple.*
   Consider the triple ST_r defined by $ST_r := (C(T), H_\alpha, D_\alpha)$, where the representation $\pi_r : C(T) \to B(H_\alpha)$ is defined via a homomorphism ϕ_r of $C(T)$ onto $\mathcal{A}_\alpha$ as follows:

   (i) $\forall f \in C(T), \forall s \in [0, \alpha]: \phi_r(f)(s) := f(r(s))$;
   (ii) $\forall f \in C(T), \pi_r(f) := \pi_\alpha(\phi_r(f))$.

Then ST_r is an unbounded Fredholm module, which is summable for any $s > 1$ and not summable for $s = 1$.

Proof. The Hilbert space, the representation and the Dirac operator are mostly inherited from the α-interval triple, so this makes it quite easy to verify the properties demanded by an unbounded Fredholm module. The only remaining problem is to prove that the subspace

$$LC := \left\{ f \in C(T) \mid \left[D_\alpha, \pi_r(f) \right] \text{ is densely defined and bounded} \right\}$$

is dense in $C(T)$. The proof of this may be based on the Stone–Weierstrass Theorem. We first remark that the Leibniz differentiation rule implies that LC is a unital self-adjoint algebra. We then only have to prove that the functions in LC separate the points of T. An argument proving this can be based on Urysohn's Lemma and Tietze's Extension Theorem. $\quad\square$

We can then associate as follows an unbounded Fredholm module to a continuous curve.

Definition 4.2. Let $r : [0, \alpha] \to T$ be a continuous injective mapping and $(\mathcal{A}_\alpha, H_\alpha, D_\alpha)$ the α-interval spectral triple. The unbounded Fredholm module $ST_r := (C(T), H_\alpha, D_\alpha)$ is then called the *unbounded Fredholm module associated to the continuous curve r*.

We close this section by computing the metric d_r on T induced by the parameterization r of $\mathcal{R}$.

Proposition 4.3. *Let $r : [0, \alpha] \to T$ be a continuous injective mapping, and $ST_r = (C(T), H_\alpha, D_\alpha)$ the unbounded Fredholm module associated to r. The metric d_r induced on T by ST_r is given by*

$$d_r(p,q) = \begin{cases} 0 & \text{if } p = q, \\ \infty & \text{if } p \neq q \text{ and } (p \notin \mathcal{R} \text{ or } q \notin \mathcal{R}), \\ |r^{-1}(p) - r^{-1}(q)| & \text{if } p \neq q \text{ and } p \in \mathcal{R} \text{ and } q \in \mathcal{R}. \end{cases}$$

Proof. If one of the points, say p, is not a point on the curve and $q \neq p$, then by Urysohn's Lemma there exists a nonnegative continuous function f on $\mathcal{T}$ such that $f(p) = 1$, $f(q) = 0$ and, for any point $r(t)$ from $\mathcal{R}$, we have $f(r(t)) = 0$. This means that $\pi_r(f) = 0$; so for any $N \in \mathbb{N}$, we have $\|[D_\alpha, \pi_r(Nf)]\| \leqslant 1$. Hence, $d_r(p,q) \geqslant N$ and $d_r(p,q) = \infty$.

Suppose now that both p and q belong to $\mathcal{R}$ and $p \neq q$. Then it follows that the homomorphism $\phi_r : C(\mathcal{T}) \to C([0,\alpha])$, as defined in Proposition 4.1, has the property that r-triple factors through the α-interval triple via the identities presented in Proposition 4.1, and we deduce that

$$\forall f \in C(\mathcal{T}): \quad \|[D_\alpha, \pi_r(f)]\| \leqslant 1 \quad \Leftrightarrow \quad \|[D_\alpha, \pi_\alpha(\phi_r(f))]\| \leqslant 1.$$

Hence, on the curve $r[0,\alpha])$, the metric induced by the spectral triple for the curve is simply the metric which r transports from the interval $[0,\alpha]$ to the curve. The proposition follows. $\quad\square$

5. Sums of curve triples

Let $\mathcal{T}$ be a compact and Hausdorff space and suppose that for $1 \leqslant i \leqslant k$, we are given continuous curves $r_i : [0,\alpha_i] \to \mathcal{T}$. It is then fairly easy to define the direct sum of the associated r_i-unbounded Fredholm modules, but the sum may fail to be an unbounded Fredholm module in the sense that there may not exist a dense set of functions which have bounded commutators with all the Dirac operators simultaneously. It turns out that if the curves do not overlap except at finitely many points, then this problem cannot occur.

Proposition 5.1. *Let $\mathcal{T}$ be a compact and Hausdorff space and for $1 \leqslant i \leqslant h$, let $r_i: [0,\alpha_i]$ be a continuous curve. If for each pair $i \neq j$ the number of points in $r_i([0,\alpha_i]) \cap r_j([0,\alpha_j])$ is finite, then the direct sum $\bigoplus_{i=1}^{h} ST_{r_i}$ is an unbounded Fredholm module for $C(\mathcal{T})$.*

Proof. For each $i \in \{1, \dots, h\}$, the r_i-spectral triple, ST_{r_i} is defined by $ST_{r_i} := (C(\mathcal{T}), H_{\alpha_i}, D_{\alpha_i})$. Further, the direct sum is defined by

$$\bigoplus_{i=1}^{h} ST_{r_i} := \left(C(\mathcal{T}), \bigoplus_{i=1}^{h} H_{\alpha_i}, \bigoplus_{i=1}^{h} D_{\alpha_i} \right).$$

In order to prove that

$$\left\| \left[\bigoplus_{i=1}^{h} D_{\alpha_i}, \left(\bigoplus_{i=1}^{h} \pi_{\alpha_i} \right)(f) \right] \right\| < \infty$$

for a dense set of continuous complex functions on $\mathcal{T}$, we will again appeal to the Stone–Weierstrass theorem, and repeat most of the arguments we used to prove that any ST_r is an unbounded Fredholm module. We therefore define a set of functions $\mathcal{A}$ by

$$\mathcal{A} := \left\{ f \in C(\mathcal{T}) \,\middle|\, \forall i, \ 1 \leqslant i \leqslant h, \left\| [D_{\alpha_i}, \pi_{\alpha_i}(f)] \right\| < \infty \right\}.$$

As before, we see that $\mathcal{A}$ is a self-adjoint unital algebra. Moreover, we find, just as in the proof of Proposition 4.1, that given two points p and q in $\mathcal{T}$ for which at least one of them does not belong to the union of the points on the curves, $\bigcup_{i=1}^{h} \mathcal{R}_i$, there is a function f in $\mathcal{A}$ such that $f(p) \neq f(q)$. Let us then suppose that both p and q are points on the curves, say $p \in \mathcal{R}_j$ and $p = r_j(t_p)$, $q \in \mathcal{R}_k$ and $q = r_k(t_q)$. We will then define a continuous function g on the compact set $\bigcup_{i=1}^{h} \mathcal{R}_i$ by first defining it on $\mathcal{R}_j$, then extending its definition to all the curves which meet $\mathcal{R}_j$, and then to all the curves which meet any of these, and so on. Finally, it is defined to be zero on the rest of the curves, which are not connected to $\mathcal{R}_j$. The definition of g on $\mathcal{R}_j$ is given in the following way. First, we define a continuous function c_j in $\mathcal{A}_{\alpha_j}$ by

$$c_j(t_p) := 1;$$

$$\text{if } 0 \neq t_p, \text{ then } c_j(0) := 0;$$

$$\text{if } t_p \neq \alpha_j, \text{ then } c_j(\alpha_j) := 0;$$

$$c_j \text{ is extended to a piecewise affine function on } [0, \alpha_j];$$

$$c_j \text{ is even on } [-\alpha_j, \alpha_j].$$

By Proposition 3.2, such a function c_j is in the domain of D_{α_j} and $D_{\alpha_j} c_j$ is a bounded function. We will then transport c_j to a continuous function g on $\mathcal{R}_j$ by

$$\forall t \in [0, \alpha_j]: \quad g\big(r_j(t)\big) := c_j(t).$$

The next steps then consist in extending g to more curves by adding one more curve at each step. In order to do this, we will give the necessary induction argument. Let us then suppose that g is already defined on the curves $\mathcal{R}_{i_1}, \ldots, \mathcal{R}_{i_n}$ and assume that there is yet a curve $\mathcal{R}_{i_{n+1}}$ on which g is not defined but has the property that $\mathcal{R}_{i_{n+1}}$ intersects at least one of the curves $\mathcal{R}_{i_1}, \ldots, \mathcal{R}_{i_n}$. Clearly, there are only finitely many points $s_1, \ldots, s_m$ on $\mathcal{R}_{i_{n+1}}$ which are also in the union of the first n curves. Suppose that these points are numbered in such a way that

$$0 \leqslant r_{i_{n+1}}^{-1}(s_1) < \cdots < r_{i_{n+1}}^{-1}(s_m) \leqslant \alpha_{i_{n+1}}.$$

We can then define a piecewise affine, continuous and even function $c_{i_{n+1}}$ on $[-\alpha_{i_{n+1}}, \alpha_{i_{n+1}}]$ by

$$\forall l \in \{1, \ldots, m\}: \quad c_{i_{n+1}}\big(r_{i_{n+1}}^{-1}(s_l)\big) := g(s_l);$$

$$\text{if } 0 \neq s_1, \text{ then } c_{i_{n+1}}(0) := 0;$$

$$\text{if } s_m \neq \alpha_{i_{n+1}}, \text{ then } c_{i_{n+1}}(\alpha_{i_{n+1}}) := 0;$$

$$c_{i_{n+1}} \text{ is extended to a piecewise affine function on } [0, \alpha_{i_{n+1}}];$$

$$c_{i_{n+1}} \text{ is even on } [-\alpha_{i_{n+1}}, \alpha_{i_{n+1}}].$$

This function is in the domain of $D_{\alpha_{i_{n+1}}}$ and $D_{\alpha_{i_{n+1}}} c_{i_{n+1}}$ is bounded. The function $c_{i_{n+1}}$ can then be lifted to $\mathcal{R}_{i_{n+1}}$ and we may define g on this curve by

$$\forall t \in [0, \alpha_{i_{n+1}}]: \quad g\big(r_{i_{n+1}}(t)\big) := c_{i_{n+1}}(t).$$

This process will stop after a finite number of steps and will yield a continuous function g on the union of the curves which are connected to p. On the other curves, g is defined to be 0, and by Tietze's extension theorem, this function can then be extended to a continuous real function on all of $\mathcal{T}$. By construction, the function g belongs to $\mathcal{A}$. Further, its restriction to the union $\bigcup \mathcal{R}_{i=1}^h$ has the property that $g(p) = 1$ and for all other points s in $\bigcup \mathcal{R}_{i=1}^h$, we have $g(s) < 1$. In particular, $g(q) \neq g(p)$. Hence, the theorem follows. $\quad\square$

6. Parameterized graphs

We will now turn to the study of *finite graphs* where each edge, say e, is equipped with a weight or rather a length, say $\ell(e)$. Such a graph, is called a *weighted graph*. This concept has occurred in many places and it is very well described in the literature on graph theory and its applications. Our point of view on this concept is related to the study of the so-called *quantum graphs*, although it was developed independently of it. We refer the reader to the survey article [23] by Peter Kuchment on this kind of weighted graphs and their properties. In Section 2 of the article by Kuchment, a quantum graph is described as a graph for which each edge e is considered as a line segment connecting two vertices. Any edge e is given a length $\ell(e)$ and then equipped with a parameterization, as described in the definition just below; it then becomes homeomorphic to the interval $[0, \ell(e)]$. This allows one to think of the quantum graph as a one-dimensional simplicial complex.

Definition 6.1. A weighted graph $\mathcal{G}$ with vertices $\mathcal{V}$, edges $\mathcal{E}$ and a length function ℓ, is said to have a *parameterization* if it is a subset of a compact and Hausdorff space $\mathcal{T}$ such that the vertices in $\mathcal{V}$ are points in $\mathcal{T}$ and for each edge e there exists a pair of vertices (p, q) and a continuous curve $\{e(t) \in \mathcal{T} \mid 0 \leqslant t \leqslant \ell(e)\}$ without self-intersections such that $e(0) = p$ and $e(\ell(e)) = q$. Two edges $e(t)$ and $f(s)$ may only intersect at some of their endpoints. Given $e \in \mathcal{E}$, the length of the defining interval, $\ell(e)$, is called the length of the edge e. The set of all points on the edges is denoted $\mathcal{P}$.

Even though the foregoing definition indirectly points at an orientation on each edge, this is not the intention. Below we will allow *traffic in both directions* on any edge, but we will only have one parameterization. One may wonder if any finite graph has a parameterization as a subset of some space $\mathbb{R}^n$, but it is quite obvious that if one places as many points as there are vertices inside the space $\mathbb{R}^3$ such that the diameter of this set is at most half of the length of the shortest edge, then the points can be connected as in the graph with smooth nonintersecting strings having the correct lengths. Consequently, it makes sense to speak of the set of points $\mathcal{P}$ of the graph and of the distances between the points in $\mathcal{P}$.

The graphs we are studying have natural representations as compact subsets of $\mathbb{R}^n$, so we would like to have this structure present when we consider the graphs. Hence, we would like to think of our graphs as embedded in some compact metric space. In the case of a finite graph, which we are considering in this section, the ambient space may be nothing but the simplicial complex, described above, and equipped with a parameterization of the 1-simplices.

Each edge e in a parameterized graph has an internal metric given by the parameterization and then, as in [23], it is possible to define the distance between any two points from the same connected component of $\mathcal{P}$ as the length of the shortest path connecting the points. In order to keep this paper self-contained, however, we will next introduce the notions of a path and length of a path in this context. We first recall the graph-theoretic concept of a path as a set of edges

$\{e_1, \ldots, e_k\}$ such that $e_i \cap e_{i+1}$ is a vertex and no vertex appears more than once in such an intersection. It is allowed that the starting point of e_1 equals the endpoint of e_k, in which case we say that the path is closed, or that it is a *cycle*. We can then extend the concept of a path to a parameterized graph as follows.

Definition 6.2. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with parameterization. Let e, f be two edges, $e(s)$ a point on e and $f(t)$ a point on f. Let $\mathcal{G} \cup \{e(s), f(t)\}$ be the parameterized graph obtained from $\mathcal{G}$ by adding the vertices $e(s), f(t)$ and, accordingly, dividing some of the edges into 2 or 3 edges. A *path* from $e(s)$ to $f(t)$ is then an ordinary graph-theoretic path in $\mathcal{G} \cup \{e(s), f(t)\}$ starting at $e(s)$ and ending at $f(t)$.

The intuitive picture is that a path from $e(s)$ to $f(t)$ starts at $e(s)$ and runs along the edge upon which $e(s)$ lies to an endpoint of that edge. From there, it continues on an ordinary graph-theoretic path to an endpoint of the edge upon which $f(t)$ is placed, and then continues on this edge to $f(t)$. This is also almost what is described in the definition above, but not exactly. The problem which may occur is that $e(s)$ and $f(t)$ actually are points on the same edge, and then we must allow the possibility that the path runs directly from $e(s)$ to $f(t)$ on this edge, as well as the possibility that the path starts from $e(s)$, runs away from $f(t)$ to an endpoint, and then finally comes back to $f(t)$ by passing over the other endpoint of the given edge.

Now that we have to our disposal the concepts of paths between points, lengths of edges and internal distances on parts of edges, the *length of a path* is simply defined as the sum of the lengths of the edges and partial edges of which the path is made of. Since the graphs we consider are finite, there are only finitely many paths between two points $e(s)$ and $f(t)$ and we can define the *geodesic distance*, $d_{\text{geo}}(e(s), f(t))$, between these two points as the minimal length of a path connecting them. If there are no paths connecting them, then the geodesic distance is defined to be infinite. Each connected component of the graph is then a compact metric space with respect to the geodesic distance, and on each component, the topology induced by the metric is the same as the one the component inherits from the ambient compact space.

A main emphasis of Kuchment's research on quantum graphs (see [23,24]) is the study of the spectral properties of a second order differential operator on a quantum graph. This is aimed at constructing mathematical models which can be applied in chemistry, physics and nanotechnology. On the other hand, our goal is to express geometric features of a graph by noncommutative geometric means. This requires, in the first place, associating a suitable spectral triple to a graph.

We have found in the literature several proposals for spectral triples associated to a graph. (See [6, Section IV.5], [10,11,33].) We agree with Requardt when he states in [33] that it is a delicate matter to call any of the spectral triples the "right one," any proposal for a spectral triple associated to a graph being in fact determined by the kind of problem one wants to use it for.

Our own proposal for a spectral triple is based on the length function associated to the edges, which, as was stated before, brings us close to the quantum graph approach. A major difference between the quantum graph approach and ours, however, is that the delicate question of which boundary conditions one has to impose in order to obtain *self-adjointness* of the basic differential operator $\frac{1}{i}\frac{d}{dx}$ on each edge disappears in our context. The reason being that our spectral triple associated with the edges takes care of that issue by introducing a larger module than just the space of square-integrable functions on the line.

We are now going to construct a spectral triple for a parameterized graph by forming a direct sum of all the unbounded Fredholm modules associated to the edges. This is a special case of the direct sum of curve triples as studied in Proposition 5.1, so we may state

Definition 6.3. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a weighted graph with a parameterization in a compact metric space $\mathcal{T}$ and let $\mathcal{P}$ denote the subset of $\mathcal{T}$ consisting of the points of $\mathcal{G}$. For each edge $e = \{e(s) \mid 0 \leqslant s \leqslant \ell(e)\}$, we let ST_e denote the e-triple. The direct sum of the ST_e-triples over $\mathcal{E}$ is an unbounded Fredholm module over $C(\mathcal{T})$ and a spectral triple for $C(\mathcal{P})$, which we call the *graph triple* of $\mathcal{G}$ and denote $ST_{\mathcal{G}}$. The metric induced by the graph triple on the set of points $\mathcal{P}$ is denoted $d_{\mathcal{G}}$.

Our next result shows that $d_{\mathcal{G}}$ coincides with the geodesic metric on the graph $\mathcal{G}$.

Proposition 6.4. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a weighted graph with a parameterization. Then, for any points $e(s)$, $f(t)$ on a pair of edges e, f, we have*

$$d_{\mathcal{G}}\big(e(s), f(t)\big) = d_{\mathrm{geo}}\big(e(s), f(t)\big).$$

Proof. We will define a set of functions $\mathcal{N}$ by

$$\mathcal{N} = \big\{k : \mathcal{P} \to \mathbb{R} \mid \exists e(s) \,\forall f(t) \colon k\big(f(t)\big) = d_{\mathrm{geo}}\big(f(t), e(s)\big)\big\}.$$

Locally, the function $f \in \mathcal{N}$ has slope 1 or minus 1 with respect to the geodesic distance. This is seen in the following way. Let f be a function in $\mathcal{N}$, and suppose you move from a point, say $f(t_0)$, on the edge f to the point $f(t_1)$ on the same edge; then the geodesic distance to $e(s)$ will usually either increase or decrease by the amount $|t_1 - t_0|$. This means that *most often* it is expected that in a neighborhood around a point t_0 the function $k(f(t))$ is given either as $k(f(t)) = k(f(t_0)) + (t - t_0)$ or $k(f(t)) = k(f(t_0)) - (t - t_0)$. On the interval $(t_0 - \delta, t_0 + \delta)$, the function $t \mapsto k(f(t))$ is then differentiable and its derivative is either constantly 1 or -1.

The reason why this picture is not always true is that for a point $f(t_0)$ on an edge f, the geodesic paths from $e(s)$ to the points $f(t_0 - \varepsilon)$ and $f(t_0 + \varepsilon)$ may use different sets of edges. The function $k(f(t))$ is still continuous at t_0, but the derivative will change sign from 1 to -1, or the other way around. For each edge f, the function k on f is then transported via the homomorphism attached to the f-triple onto the function $\phi_f(k) \in \mathcal{A}_{l(f)}$ and further, via the homomorphism $\Phi_{l(f)}$, onto the continuous and even function $\Phi_{l(f)}(\phi_f(k))$ on $[-\ell(f), \ell(f)]$ on the positive part of the interval. The function $\Phi_{l(f)}(\phi_f(k))$ is also piecewise affine with slopes in the set $\{-1, 1\}$. From Proposition 3.2, we then deduce that $D_{\ell(f)}\Phi_{l(f)}(\phi_f(k))$ exists and is essentially bounded by 1. Given this fact, it follows that for all functions k in $\mathcal{N}$ and any edge f we have $\|[D_{\ell(f)}, \pi_f(k)]\| \leqslant 1$, so with the above choice of function k, we obtain

$$\forall e, f \in \mathcal{E}, \ \forall t \in \big[0, \ell(f)\big], \ \forall s \in \big[0, \ell(e)\big]:$$
$$d_{\mathrm{geo}}\big(f(t), e(s)\big) = k\big(f(t)\big) - k\big(e(s)\big) \leqslant d_{\mathcal{G}}\big(f(t), e(s)\big).$$

Suppose now that we are given points as before, $e(s)$ and $f(t)$, on some edges e and f. Then there exists a geodesic path from $e(s)$ to $f(t)$, and we may without loss of generality assume that this path is an ordinary graph-theoretic path consisting of the edges $e_1, \ldots, e_k$ such that the starting point of e_1 is $e(s)$ and the endpoint of e_k is $f(t)$. Any continuous function g on $\mathcal{P}$ which has the property that

$$\max_{e \in \mathcal{E}} \big\|\big[D_{\ell(e)}, \pi_e(g)\big]\big\| \leqslant 1$$

also has the property that its derivative on each edge is essentially bounded by 1. In particular, this means that for each edge e_j, with $1 \leqslant j \leqslant k$, we must have $|g(e_j(\ell(e_j))) - g(e_j(0))| \leqslant \ell(e_j)$ and then, since $e_i(\ell(e_i)) = e_{i+1}(0)$ for $1 \leqslant i \leqslant k - 1$, we have

$$
\begin{aligned}
\left| g\big(e(s)\big) - g\big(f(t)\big) \right| &= \left| g\big(e_1(0)\big) - g\big(e_k(\ell(e_k))\big) \right| \\
&\leqslant \sum_{j=1}^{k} \left| g\big(e_j(0)\big) - g\big(e_j(\ell(e_j))\big) \right| \\
&\leqslant \sum_{j=1}^{k} \ell(e_j) \\
&= d_{\mathrm{geo}}\big(e(s), f(t)\big).
\end{aligned}
$$

Since this holds for any such function g, we deduce that

$$
d_{\mathcal{G}}\big(e(s), f(t)\big) \leqslant d_{\mathrm{geo}}\big(e(s), f(t)\big),
$$

and the proposition follows. $\square$

It should be remarked that the unbounded Fredholm modules of noncommutative geometry can be refined to give more information about the topological structure of the graph. Suppose for instance that the graph is connected but is not a tree. It then contains at least one closed path, *a cycle*. For each cycle, one can obtain a parameterization by adding the given parameterizations or the inverses of the given parameterizations of the edges which go into the cycle. In this way, the cycle can be described via a spectral triple for a circle of length equal to the sum of the edges used in the cycle. If this unbounded Fredholm module is added to the direct sum of the curve-triples coming from each edge, then the geodesic distance is still measured by the new spectral triple, and this triple will induce an element in the K-homology of the graph, as in [6, Chapter IV, Section 8.δ]. This element in the K-homology group will be able to measure the winding number of a nonzero continuous function around this cycle. One may, of course, take one such a summand for each cycle and in this way obtain an unbounded Fredholm module which keeps track of the connectedness type of the graph. We have not yet found a suitable use for this observation in the case of finite graphs, but we would like to pursue this idea in connection with our later study of the Sierpinski gasket in Section 8. (See, especially, item (vi) in Section 9, along with the relevant discussion following it.)

7. Infinite trees

In the first place, it is not possible to create unbounded Fredholm modules for infinite graphs by taking a direct sum of the unbounded Fredholm modules corresponding to each of the edges and each of the cycles. The problem is that there may be an infinite number of edges of length bigger than some $\delta > 0$. In such a case, the direct sum of the Dirac operators will not have a compact resolvent any more. We will avoid this problem by only considering graphs which we call *finitely summable trees* and which we define below, but before that, we want to remark that the concept of a path remains the same, namely, a finite collection of edges leading from one vertex to another without repetitions of vertices.

Definition 7.1. An infinite graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a *finitely summable tree* if the following conditions are satisfied:

 (i) There are at most countably many edges.
 (ii) There exists a length function ℓ on $\mathcal{E}$ such that for any edge e, we have $\ell(e) > 0$.
 (iii) For any two vertices u, v from $\mathcal{V}$, there exists exactly one path between them.
 (iv) There exists a real number $p \geqslant 1$ such that $\sum_{e \in \mathcal{E}} \ell(e)^p < \infty$. (In that case, the tree is said to be *p-summable*.)

An infinite tree which is not finitely summable may create problems when one tries to look at it in the same way as we did for finite graphs. It may, for instance, not be possible to embed such a graph into a locally compact space in a reasonable way. To indicate what the problems may be, we now discuss an example. Think of the bounded tree whose vertices v_n are indexed by $\mathbb{N}_0$ and whose edges all have length 1 and are given by $e_n := \{\{v_0, v_n\} \mid n \in \mathbb{N}\}$, i.e. all the edges go out from v_0 and the other vertices are all endpoints, and they are all one unit of length away. If the points v_n for $n \geqslant 1$ have to be distributed in a symmetric way in a metric space, then the distance between any two of them should be the same and no subsequence of this sequence will be convergent. This shows right away that the graph cannot be embedded in a compact space in a reasonable way, but it also shows that the point v_0 does not have a compact neighborhood, so even an embedding in a locally compact space is not possible. The restriction of working with finitely summable trees will be shown to be sufficient in order to embed the graph in a compact metric space. Before we show this, we would like to mention that the point of view of this article is to consider edges as line segments rather than as pairs of vertices. We will start, however, from the graph-theoretic concept of edges as pairs of vertices and then show very concretely that we can obtain a model of the corresponding simplicial complex, even in such a way that the edges are continuous curves in a compact metric space.

Definition 7.2. A finitely summable tree $\mathcal{G}$ with vertices $\mathcal{V}$ and edges $\mathcal{E}$ is said to have a *parameterization* if $\mathcal{V}$ can be represented injectively as a subset of a metric space $(\mathcal{T}, d)$ and for each edge $e = \{p, q\}$, there exists a continuous curve, $\{e(t) \in \mathcal{T} \mid 0 \leqslant t \leqslant \ell(e)\}$, without self-intersections and leading from one of the vertices of e to the other. Two curves $e(t)$ and $f(s)$ for different edges e and f may only intersect at endpoints. The length of the defining interval, $\ell(e)$, for a curve $e(t)$, is called the *length of the curve*. The set of points $e(t)$ on all the curves is denoted $\mathcal{P}$.

For an infinite tree, there is only one path between any two vertices in $\mathcal{V}$. This is the basis for the following concrete parameterization of an infinite p-summable tree inside a compact subset of the Banach space $\ell^p(\mathcal{E})$. We first fix a vertex u in $\mathcal{V}$ and map this to 0 in $\ell^p(\mathcal{E})$; then the unique path from u to any other vertex v determines the embedding in a canonical way. In order to describe this construction, we will denote by δ_e the canonical unit basis vector in $\ell^p(\mathcal{E})$ corresponding to an edge e in $\mathcal{E}$.

Proposition 7.3. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a finitely summable tree, u a vertex in $\mathcal{V}$ and $T_u : \mathcal{V} \to \ell^p(\mathcal{E})$ be defined by*

$$T_u(w) := \begin{cases} 0 & \text{if } w = u, \\ \sum_{j=1}^{k} \ell(e_j)\delta_{e_j} & \text{if } w \neq u \text{ and the path is } \{e_1, \ldots, e_k\}. \end{cases}$$

For an arbitrary edge $e = \{w_1, w_2\}$, we choose an orientation such that the first vertex is the one which is nearest to u. Let us suppose that w_1 is the first vertex. Then, given such an oriented edge $e = (w_1, w_2)$, a curve $e_u : [0, \ell(e)] \to \ell^p(\mathcal{E})$ is defined by

$$e_u(t) := T_u(w_1) + t\delta_e.$$

Let $\mathcal{P}_u$ denote the set of points on all the curves $e_u(t)$ and let $\mathcal{T}_u$ denote the closure of $\mathcal{P}_u$. Then $\mathcal{T}_u$ is a norm compact subset of $\ell^p(\mathcal{E})$ and the triple $(\mathcal{T}_u, T_u, \{e_u(t) \mid e \in \mathcal{E}\})$ constitutes a parameterization of $\mathcal{G}$, in the sense of Definition 7.2.

Proof. The only thing which really has to be proved is that the set $\mathcal{T}_u$ is a norm compact subset of $\ell^p(\mathcal{E})$. Let then $\varepsilon > 0$ be given and choose a finite and connected subgraph $\mathcal{G}_0 = (\mathcal{V}_0, \mathcal{E}_0)$ of $\mathcal{G}$ such that $\sum_{e \in \mathcal{E}_0} \ell(e)^p + \varepsilon^p > \sum_{e \in \mathcal{E}} \ell(e)^p$ and u is a vertex in $\mathcal{V}_0$. We denote by $\mathcal{P}_{u, \mathcal{E}_0}$ the set of points on all the curves $e_u(t)$, where $e \in \mathcal{E}_0$. We then remark that $\mathcal{P}_{u, \mathcal{E}_0}$ is just a finite union of compact sets, and is therefore compact. By construction, any point in $\mathcal{P}_u$ is within a distance ε of $\mathcal{P}_{u, \mathcal{E}_0}$, so the closure $\mathcal{T}_u$ of $\mathcal{P}_u$ is compact. $\quad\square$

It is not difficult to see that for two different vertices u and v, the metric spaces $\mathcal{T}_u$ and $\mathcal{T}_v$ obtained via the above construction are actually isometrically isomorphic via the rather natural identification of the sets $\mathcal{P}_u$ and $\mathcal{P}_v$ described just below.

Proposition 7.4. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a finitely summable tree. Consider u and v two different vertices in $\mathcal{V}$ and let $T_u : \mathcal{V} \to \ell^p(\mathcal{E})$, respectively $T_v : \mathcal{V} \to \ell^p(\mathcal{E})$, be defined as in Proposition 7.3. Then the mapping $S_{uv} : \mathcal{P}_u \to \mathcal{P}_v$ defined by*

$$S_{uv}(x) := \begin{cases} T_v \circ T_u^{-1}(x) & \text{if } x \in T_u(\mathcal{V}), \\ T_v(w_j) + t\delta_e & \text{if } x = e_u(t), \ e = (w_1, w_2), \ t \in (0, l(e)), \end{cases}$$

where w_j is the nearest vertex to v among w_1 and w_2, is an isometric isomorphism of $\mathcal{P}_u$ onto $\mathcal{P}_v$.

Proof. Given two points, say x, y, in $\mathcal{P}_u$, we have to show that

$$d\bigl(S_{uv}(x), S_{uv}(y)\bigr) = d(x, y).$$

There has to be two different arguments showing this, according to the cases where x and y are on the same or on different edges. We will only consider the case where $x = e_u(s)$ on an edge e and $y = g_u(t)$ on a different edge g. Then there exists a finite set of vertices $w_1, \ldots, w_k$ such that the closed line segments in $\ell^p(\mathcal{E})$ given by

$$\bigl[e_u(s), T_u(w_1)\bigr], \ \bigl[T_u(w_1), T_u(w_2)\bigr], \ \ldots, \ \bigl[T_u(w_{k-1}), T_u(w_k)\bigr], \ \bigl[T_u(w_k), g_u(t)\bigr],$$

all belong to $\mathcal{P}_u$ and constitute the unique path herein from x to y. The distance in $\mathcal{P}_u$ from x to y is given by

$$\|x - y\|_p = \left(\bigl\|T_u(w_1) - e_u(s)\bigr\|^p + \bigl\|g_u(t) - T_u(w_k)\bigr\|^p + \sum_{i=1}^{k-1} \ell\bigl(\{w_i, w_{i+1}\}\bigr)^p \right)^{1/p}.$$

The term $\|T_u(w_1) - e_u(s)\|$ is either s or $\ell(e) - s$. It is s if w_1 is closer to u than x, and $\ell(e) - s$ otherwise. The distance in $\mathcal{P}_v$ from $S_{uv}(x)$ to $S_{uv}(y)$ is given by

$$\|S_{uv}(x) - S_{uv}(y)\|_p$$
$$= \left(\|T_v(w_1) - e_v(s')\|^p + \|g_v(t') - T_v(w_k)\|^p + \sum_{i=1}^{k-1} \ell\big(\{w_i, w_{i+1}\}\big)^p \right)^{1/p}.$$

A moment's reflection will make it clear to the reader that the distance between $S_{uv}(x)$ and $S_{uv}(y)$ is the same as the distance between x and y. Thus S_{uv} defines an isometry between the sets of points $\mathcal{P}_u$ and $\mathcal{P}_v$. $\quad\square$

Since the sets $\mathcal{P}_u$ and $\mathcal{P}_v$ are dense in $\mathcal{T}_u$ and $\mathcal{T}_v$, respectively, we see that S_{uv} extends to a natural isometry between $\mathcal{T}_u$ and $\mathcal{T}_v$. The compact metric spaces $\mathcal{T}_u$ and $\mathcal{T}_v$ are then isometrically isomorphic via an isometry which commutes with the parameterizations, so we may introduce

Definition 7.5. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a p-summable infinite tree, u a vertex in $\mathcal{V}$, and $(\mathcal{T}_u, T_u, \{e_u(t) \mid e \in \mathcal{E}\})$ the parameterization of $\mathcal{G}$ introduced in Proposition 7.3. In view of Proposition 7.4, we may use the simpler notations $\mathcal{T}$, respectively $\mathcal{P}$, to denote the sets $\mathcal{T}_u$ and $\mathcal{P}_u$. (In the sequel, we will also use the notation $\mathcal{T}^p, \mathcal{P}^p$ when needed.) The parameterization is called the *p-parameterization* of $\mathcal{G}$. The metric given by $\|\cdot\|_p$ is denoted by d_p and the complement of $\mathcal{P}$ in $\mathcal{T}$ is denoted by $\mathcal{B}$ and called the boundary.

It is clear that if an infinite tree is p-summable, then it is also q-summable for any real number $q > p$. A natural question is, of course, if the p- and q-parameterizations are homeomorphic or even Lipschitz equivalent as metric spaces. We can easily show that the spaces are homeomorphic but the Lipschitz equivalence may not be automatic, although it is easy to establish in many concrete examples. We have found a condition which ensures Lipschitz equivalence; it is nearly a tautology, but rather handy for the study of concrete examples. To be able to express this property, we first observe that the p-parameterization has a concrete realization via a base vertex u as $\mathcal{T}_u$. This is a subset of $\ell^p(\mathcal{E})$ and since the p-norm here dominates the ∞-norm, we can use this norm to introduce a metric d_∞ on $\mathcal{T}_u$ by $d_\infty(x, y) := \|x - y\|_\infty$. If one goes back to the description of $\mathcal{T}_u$, one can realize that $d_\infty(x, y)$ is independent of u and, in fact, depends only on the unique path (may be even infinite in both directions) which leads from x to y.

Proposition 7.6. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a p-summable infinite tree, and q be a real number such that $q > p$. Further, let $\mathcal{T}^p, \mathcal{T}^q$ denote the p-, respectively, q-parameterization compact metric spaces of $\mathcal{G}$. Then these spaces are homeomorphic via the natural embedding of $\mathcal{T}_p$ into $\mathcal{T}_q$.*

Moreover, the metric spaces $\mathcal{T}^p$ and $\mathcal{T}^q$ are Lipschitz equivalent if there exists a constant $k > 0$ such that

$$\forall x, y \in \mathcal{T}^p: \quad d_p(x, y) \leqslant k d_\infty(x, y).$$

Proof. Let us take a base vertex u and consider the concrete representations of $\mathcal{T}^p$ and $\mathcal{T}^q$ as $\mathcal{T}_u^p$ and $\mathcal{T}_u^g$. Since $p < q$, we have $\ell^p(\mathcal{E}) \subset \ell^q(\mathcal{E})$. Let ι denote the canonical embedding; then ι is a contraction and consequently, $\iota(\mathcal{T}_u^p)$ is a compact subset of $\mathcal{T}_u^q$ containing the point set $\mathcal{P}_u^q$

of $\mathcal{T}_u^q$. Hence, ι induces a homeomorphism between the two compact spaces $\mathcal{T}_u^p$ and $\mathcal{T}_u^q$. Let us now assume that

$$\forall x, y \in \mathcal{T}_u^p: \quad d_p(x, y) \leqslant k d_\infty(x, y),$$

and continue to work inside $\mathcal{T}_u^p$. Then we get

$$\forall x, y \in \mathcal{T}_u^p: \quad d_q(x, y) \leqslant d_p(x, y) \leqslant k d_\infty(x, y) \leqslant k d_q(x, y),$$

so the metrics are Lipschitz equivalent. $\quad\square$

It took us quite some time to realize how different various parameterizations may be. Later in this section, the reader will see (in Fig. 3) a picture of the fractal which is usually associated to the free noncommutative group on 2 generators. In Connes' book [6], on page 341, one can see quite a different picture. In the first case, the boundary is totally disconnected, whereas in the Poincare disk picture used in [6], the boundary is the unit circle. The p-parameterization has the property that it separates different boundary points very much.

Proposition 7.7. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an infinite p-summable tree and $\mathcal{T}$ its p-parameterization space. The set $\mathcal{B}$ of boundary points of $\mathcal{P}$ in $\mathcal{T}$ is closed and totally disconnected.*

Proof. Let a and b be two different points in $\mathcal{B}$ and let δ be a positive number less than $d(a, b)/3$. To δ we associate a finite connected subgraph $\mathcal{G}_0 = (\mathcal{V}_0, \mathcal{E}_0)$ such that

$$\sum_{e \in \mathcal{E}_0} \ell(e)^p + \delta^p > \sum_{e \in \mathcal{E}} \ell(e)^p.$$

The graph $\mathcal{G}_0$ is a finite tree, so it has at least two ends, i.e. vertices of degree 1. By examination of the concrete space $\mathcal{T}_u$, one can easily see that the set of points $\mathcal{P}_0$ of $\mathcal{G}_0$ has nonempty interior as a subset of $\mathcal{T}$, and that this interior, say $\mathring{\mathcal{P}}_0$, is exactly all of $\mathcal{P}_0$ except its endpoints. Given a point x in $\mathcal{P}$ which is in the complement, say $\mathcal{C}$, of $\mathring{\mathcal{P}}_0$ in $\mathcal{T}$, we see that there must be a path from x to $\mathcal{P}_0$ which meets $\mathcal{P}_0$ at an endpoint. This means that all the points in $\mathcal{C}$ are grouped into a finite number of pairwise disjoint sets according to which end vertex in $\mathcal{V}_0$ is the nearest. Suppose now that x and y are points in $\mathcal{C}$ such that their nearest end vertices in $\mathcal{V}_0$, say v and w, respectively, are different. Then, by the construction of the metric space $(\mathcal{T}, d)$, one finds that $d(x, y) > d(v, w) > 0$. This shows that the connected components of $\mathcal{C}$ can be labeled by the end vertices of $\mathcal{V}_0$ as $\mathcal{C}_v$ and there is a positive distance between any two different components.

Let us now return to the boundary points a, b and show that they fall in different components $\mathcal{C}_u$ and $\mathcal{C}_v$. Suppose to the contrary that both a and b belong to the same component, say $\mathcal{C}_u$. Then there exist vertices v and w in $\mathcal{C}_u \cap \mathcal{V}$ such that $d(a, v) < \delta$ and $d(b, w) < \delta$. Since v and w are in the same component $\mathcal{C}_u$, the unique path between them must be entirely in $\mathcal{C}_u$. This means that it uses none of the edges from $\mathcal{E}_0$; so, by the construction of $\mathcal{G}_0$, we get $d(u, v) < \delta$ and then $d(a, b) < 3\delta < d(a, b)$, a contradiction. Since the components induce a covering of the boundary by open sets, we deduce that a and b are in different components of the boundary and thus that the boundary is totally disconnected. The closedness of $\mathcal{B}$ follows, as we can take an increasing sequence of finite connected subgraphs of $\mathcal{G}$, say $(\mathcal{G}_n)$, such that the union of the sequence of open sets $(\mathring{\mathcal{P}}_n)$ in $\mathcal{T}$ is all of $\mathcal{P}$. $\quad\square$

We are not going to study the properties of the p-parameterization in more details since our main interest is to describe certain aspects of graphs with the help of noncommutative geometry. The possibility of embedding a graph as a dense subset of a compact metric space $\mathcal{T}$ shows that we can study this space through a spectral triple which is the sum of triples associated the edges of the tree. The ambient metric space has a metric which is constructed such that the space becomes compact. On the other hand the natural distance between points on a tree is the length of the unique path between the points and this will quite often not be a bounded metric on the tree, so we will refer to this distance as the geodesic distance.

Definition 7.8. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph which is embedded in a metric space $\mathcal{T}$ such that each edge, $e = (x_0, x_1)$, where x_0 and x_1 are vertices in $\mathcal{V}$, has a parameterization $r_e(t)$ with $r_e(0) = x_0$ and $r_e(\ell(e)) = x_1$. Then the geodesic distance between any two points $x, y \in \mathcal{T}$, which are situated on edges on the graph, is defined to be the infimum of the sums of the lengths of the corresponding intervals which have to be used in order to construct a curve from x to y based on the parameterization given for each edge.

The p-parameterization provides a framework which makes it possible to obtain a spectral triple as a sum of triples based on curves, where the curves are parameterized edges of the tree. We will concentrate our investigations on the spectral triple which can now be obtained in the same way as was done for finite graphs. Before we embark on this, we would like to mention that unless the summability number p equals 1, one should not expect that the geodesic distance on $\mathcal{P}$ is bounded. Think, for instance, of a tree embedded in $\mathbb{R}_+$ with vertices (v_n) indexed by the natural numbers, edges of the form $\{v_n, v_{n+1}\}$ and lengths $\ell(\{v_n, v_{n+1}\}) = 1/n$. This graph is 2-summable and the set of path lengths is unbounded. Let us further remark that, by applying the triangle inequality, one can see that the geodesic distance is always larger than the distance on the p-parameterization space, which is a subset of $\ell^p(\mathcal{E})$. Thus it may very well be that the geodesic distance is unbounded on $\mathcal{P}$ and hence, in such a case, is not extendable to a continuous function on the compact space $\mathcal{T}$.

In [6, Section IV.5], *Fredholm Modules and Rank-One Discrete Groups*, Connes studies modules which are associated to trees. His modules are ℓ^2-spaces over sets consisting of points and edges, whereas the ones we are going to construct are sums of L^2-spaces over the edges, as described for finite graphs in Section 6. There may be closer relations than we can see right now between the two types of modules; at least, it seems that Proposition 6 on page 344 of [6] is related to our Theorem 7.10.

Definition 7.9. Let $\mathcal{T}$ be a compact space, $r : [0, \alpha] \to \mathcal{T}$ a continuous curve, and let a be a real number. The r, a-triple

$$\big(\mathrm{C}(\mathcal{T}), H_\alpha, D_\alpha^a\big)$$

is defined as a translated unbounded Fredholm module (in the sense of Definition 2.2) of the r-triple defined in Definition 4.2. The operator D_α^a is the translate of D_α given by $D_\alpha + aI$.

Since for any $f \in \mathrm{C}(\mathcal{T})$ we have $[D_\alpha + aI, \pi_\alpha(f)] = [D_\alpha, \pi_\alpha(f)]$, the change in the Dirac operator does not affect the spectral triple much, but the eigenvalues of the Dirac operator all get translated by the number a.

Theorem 7.10. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a p-summable infinite tree with p-parameterization in the compact metric space $\mathcal{T}$. The direct sum over all the edges e from $\mathcal{E}$ of the unbounded Fredholm modules*

$$\left(\mathrm{C}(\mathcal{T}), H_{\ell(e)}, D_{\ell(e)}^{\pi/(2\ell(e))}\right)$$

is a spectral triple for $\mathrm{C}(\mathcal{T})$, which is denoted $ST_{\mathcal{G}} := (\mathrm{C}(\mathcal{T}), H_{\mathcal{G}}, D_{\mathcal{G}})$. It can only be a finitely summable module for a real number $s > 1$. Further, for a given $s > 1$, it is finitely summable if and only if

$$\sum_{e \in \mathcal{E}} \ell(e)^s < \infty.$$

Moreover, the metric $d_{\mathcal{G}}$ induced by $ST_{\mathcal{G}}$ on the points $\mathcal{P}$ of the infinite tree $\mathcal{G}$ is the geodesic distance.

Proof. The proof relies in many ways on the corresponding proof for finite graphs. For points from the set $\mathcal{P}$, we shall see that the previous arguments can be reused to a large extent. The real problem occurs when we have to deal with the boundary points $\mathcal{B}$ in $\mathcal{T}$, i.e. the set of points $\mathcal{T}$ which are not in $\mathcal{P}$. We noticed above that the geodesic distance may not be extended to a continuous function on $\mathcal{T}$, so we cannot just copy the proof of Proposition 6.4 and define the set $\mathcal{N}$ similarly. Instead, we consider the set, say $\mathcal{F}_{\mathcal{G}}$, of finite connected subgraphs $\mathcal{G}_0 = (\mathcal{V}_0, \mathcal{E}_0)$ of $\mathcal{G}$. In the proof of Proposition 7.7, we saw that the complement of the open set $\mathring{\mathcal{P}}_0$ in $\mathcal{T}$ consists of a finite collection of pairwise disjoint closed sets $\mathcal{C}_v$ labeled by the endpoints of $\mathcal{G}_0$. This makes it possible to define a dense algebra of continuous functions on $\mathcal{T}$ which will have uniformly bounded commutators for all of our unbounded Fredholm modules associated with the edges. We simply define the set of functions $\mathcal{N}$ by looking at functions which have bounded commutators for the unbounded Fredholm module of Definition 6.3 applied to some $\mathcal{G}_0$ in $\mathcal{F}_{\mathcal{G}}$ and are constant on each of the components, outside $\mathring{\mathcal{P}}_0$, i.e. the value of such a function f on a component $\mathcal{C}_v$ is $f(v)$. Any function f in $\mathcal{N}$ must have a bounded commutator with $D_{\mathcal{G}}$, since the commutator is zero except at edges in a subgraph $\mathcal{G}_0$, and here it is supposed to have bounded commutator with the Dirac operator associated to $\mathcal{G}_0$. The functions in $\mathcal{N}$ constitute a self-adjoint algebra of continuous functions on $\mathcal{T}$ since the set $\mathcal{F}_{\mathcal{G}}$ is upwards directed under inclusion. Ordinary points in $\mathcal{P}$ can be separated by functions from $\mathcal{N}$ in the same way as this was done in the proof of Proposition 5.1. For a boundary point b and an ordinary point p in $\mathcal{P}$, it will always be possible to find a graph $\mathcal{G}_0$ in $\mathcal{F}_{\mathcal{G}}$ such that p is an inner point in the set of points $\mathcal{P}_0$ associated to $\mathcal{G}_0$. The function k, which on $\mathcal{P}_0$ is defined by $k(x) := d_{\mathrm{geo}}(x, p)$ and continued by constancy on the components associated to the end vertices of $\mathcal{G}_0$, belongs to $\mathcal{N}$ and separates p and the boundary point b. The reason being that p is assumed to be an inner point in $\mathcal{P}_0$, so it will have positive geodesic distance to any end vertex of $\mathcal{G}_0$. For two different boundary points b and c, one may use the proof of Proposition 7.7 to obtain a subgraph $\mathcal{G}_0$ in $\mathcal{F}_{\mathcal{G}}$ such that b and c fall in different components. Suppose b is in $\mathcal{C}_v$ and c is in $\mathcal{C}_w$ for end vertices v and w of $\mathcal{G}_0$. As seen above, there is a function f in $\mathcal{N}$ such that $f(v) \neq f(w)$; but $f(b) = f(v)$ and $f(c) = f(w)$, so f separates b and c.

In order to prove that we have a spectral triple for $\mathrm{C}(\mathcal{T})$, we then have to show that the resolvents of $D_{\mathcal{G}}$ are compact, when bounded. This will be proven below, but first we will show that the metric induced by the set $\{f \in \mathrm{C}(\mathcal{T}) \mid \|[D_{\mathcal{G}}, \pi(f)]\| \leqslant 1\}$ is the geodesic distance on $\mathcal{P}$.

We will then return to the analogous problem for finite graphs in the proof of Proposition 6.4. Again, we show that for any two points a and b from $\mathcal{P}$, we may find a connected subgraph $\mathcal{G}_0$ in $\mathcal{F}_\mathcal{G}$ such that its set of points in $\mathcal{T}$ contain both a and b. We then conclude as in the *finite* case that $d_\mathcal{G}(a,b) = d_{\text{geo}}(a,b)$.

In order to see that we have an unbounded Fredholm module and prove the summability statement, we have to look at the eigenvalues of

$$\bigoplus_{e \in \mathcal{E}} D_{\ell(e)}^{\pi/(2\ell(e))}.$$

The eigenvalues for each summand form the set $\{(k + 1/2)\pi/\ell(e) \mid k \in \mathbb{Z}\}$; so $(D_\mathcal{G}^2 + I)^{-1}$ has the following doubly indexed set of eigenvalues

$$\left(\frac{4\ell(e)^2}{(2k+1)^2\pi^2 + 4\ell(e)^2} \right)_{(k \in \mathbb{Z}, e \in \mathcal{E})}.$$

Remark that the same eigenvalue may occur, for different edges, but only a finite number of times. Since

$$\frac{4\ell(e)^2}{(2k+1)^2\pi^2 + 4\ell(e)^2} \leqslant \frac{4\ell(e)^2}{(2k+1)^2}$$

and $\sum_{e \in \mathcal{E}} \ell(e)^s < \infty$, we see that $(D_\mathcal{G}^2 + I)^{-1}$ is compact.

With respect to the summability, we consider for a real number $s > 0$ the sum

$$\sum_{e \in \mathcal{E}} \sum_{k \in \mathbb{Z}} \left(\frac{4\ell(e)^2}{(2k+1)^2\pi^2 + 4\ell(e)^2} \right)^{s/2}.$$

The term $(2k+1)^2$ in the denominator implies that we must have $s > 1$ in order to obtain a finite sum. Now, for $s > 1$, we rewrite the double sum as follows:

$$2^s \cdot \sum_{e \in \mathcal{E}} \ell(e)^s \cdot \sum_{k \in \mathbb{Z}} \left(\frac{1}{(2k+1)^2\pi^2 + 4\ell(e)^2} \right)^{s/2}$$

$$= 2^{s+1} \cdot \sum_{e \in \mathcal{E}} \ell(e)^s \cdot \sum_{k \in \mathbb{N}_0} \left(\frac{1}{(2k+1)^2\pi^2 + 4\ell(e)^2} \right)^{s/2}.$$

For each $e \in \mathcal{E}$, there is a $k_e \in \mathbb{N}_0$ such that $(2k+1)\pi > 2\ell(e)$ for every $k > k_e$. Then, for each $e \in \mathcal{E}$, we obtain that

$$\sum_{k=0}^{k_e} \left(\frac{1}{(2k+1)^2\pi^2 + 4\ell(e)^2} \right)^{s/2} + \left(\frac{1}{2} \right)^{s/2} \sum_{k=k_e+1}^{\infty} \left(\frac{1}{(2k+1)^2\pi^2} \right)^{s/2}$$

$$\leqslant \sum_{k \in \mathbb{N}_0} \left(\frac{1}{(2k+1)^2\pi^2 + 4\ell(e)^2} \right)^{s/2} \leqslant \sum_{k \in \mathbb{N}_0} \left(\frac{1}{(2k+1)\pi} \right)^s$$

and hence it follows that the module is s-summable if and only if $s > 1$ and the sum $\sum_{e \in \mathcal{E}} \ell(e)^s$ is finite. $\square$

Example 7.11. The Cayley graph for the noncommutative free group on 2 generators $\mathbb{F}_2$.

There is a nice description of this graph as a fractal. We start with the neutral element e at the origin of $\mathbb{R}^2$. Then the generators $\{a, b, a^{-1}, b^{-1}\}$ are placed on the axes at the points

$$\{(1/2, 0), (0, 1/2), (-1/2, 0), (0, -1/2)\}.$$

Traveling right along an edge represents multiplying on the right by a, while traveling up corresponds to multiplying by b. Each new edge is drawn at half size of the previous one to give a fractal image. We start by illustrating in Fig. 2 how all words, say CGa, which begin with an a are positioned on the tree.

Figure 3 shows the entire graph which consists of four identical fractal images. Each one, say $CG_a, CG_{a^{-1}}, CG_b, CG_{b^{-1}}$, represents all the words starting with a, a^{-1}, b or b^{-1}.

This forms an infinite tree with 4 edges of length $1/2$, 12 edges of length $1/4$, 36 edges of length $1/8$ and, generally, $4 \cdot 3^{n-1}$ edges of length 2^{-n}. The sum $\sum_{e \in \mathcal{E}} \ell(e)^s$ from Theorem 7.10 can then be written as

$$\sum_{e \in \mathcal{E}} \ell(e)^s = \sum_{m=1}^{\infty} 4 \cdot 3^{m-1} 2^{-ms} = \frac{4}{3} \sum_{m=1}^{\infty} \left(3 \cdot 2^{-s}\right)^m.$$

So the module is finitely summable if and only if $s > \log 3 / \log 2$. Hence, the spectral triple of $\mathbb{F}_2$ has metric dimension $\log 3 / \log 2$, and this is exactly the Minkowski and also the Hausdorff dimension of the closure of the Cayley graph of $\mathbb{F}_2$.

As was seen just above, given any real number $p > \log 3 / \log 2$, the graph is p-summable; so we can consider its p-parameterization $\mathcal{T}^p$. Since the lengths of the edges decrease geometrically like 2^{-n}, it follows that we have

$$\forall x, y \in \mathcal{T}^p: \quad d_p(x, y) \leqslant 2 d_\infty(x, y).$$

Hence, by Proposition 7.6, $\mathcal{T}^q$ and $\mathcal{T}^p$ are Lipschitz equivalent for any $q > p > \log 3 / \log 2$. Further, some computations with the closure of the Cayley graph for $\mathbb{F}_2$ constructed above show
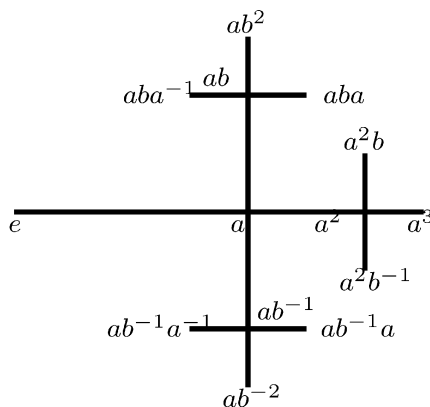
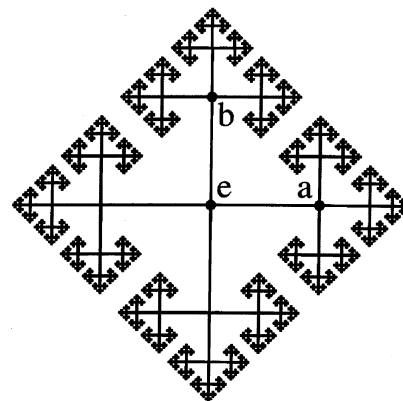Fig. 2. A portion of the Cayley graph of $\mathbb{F}_2$.   Fig. 3. The Cayley graph of $\mathbb{F}_2$, viewed as a fractal tree.

that the metric coming from $\mathbb{R}^2$ on this set is also Lipschitz equivalent to the d_∞ metric. Therefore, the $\mathbb{R}^2$ fractal is actually Lipschitz equivalent to any of the $\mathcal{T}^p$ parameterization spaces for $p > \log 3 / \log 2$.

On page 341 of [6], one can find a representation of the Cayley graph of $\mathbb{F}_2$ as a subset of the Poincare disc, and as we mentioned above, the boundaries of these two parameterizations of the Cayley graph of $\mathbb{F}_2$ are very different.

7.1. Complex dimensions of trees

The mathematical theory of complex fractal dimensions finds its origins in the study of the geometry and spectra of fractal drums [25,27] and, in particular, of fractal strings [31,32]. In the latter case, it is developed in the research monograph [29] and significantly further expanded in the recent book [30]. We establish here some connections between this theory and our work.

Definition 7.12. Let p be a real number greater than 1 and $\mathcal{G}$ an infinite p-summable tree with vertices $\mathcal{V}$ and a countable number of parameterized edges $\mathcal{E} = \{e_n \mid n \in \mathbb{N}\}$. Assume that the lengths of the edges $\ell(e_n)$ converges to 0 as $n \to \infty$. The *zeta function* of the tree $\mathcal{G}$ is denoted $\zeta_\mathcal{G}(z)$ and is defined for $\mathrm{Re}(z) > p$ by

$$\zeta_\mathcal{G}(z) = \mathrm{tr}\big(|D_\mathcal{G}|^{-z}\big).$$

In view of Theorem 7.10, $|D_\mathcal{G}|$ has the following doubly indexed set of eigenvalues.

$$\left(\frac{(2k+1)\pi}{2\ell(e)}\right)_{(k\in\mathbb{N}_0, e\in\mathcal{E})}$$

each with multiplicity 2.

Let $\zeta(z)$ denote the Riemann zeta function. By writing

$$\zeta(z) = \sum_{l=1}^{\infty} l^{-z} = \sum_{k=0}^{\infty}(2k+1)^{-z} + \sum_{k=1}^{\infty}(2k)^{-z} = \sum_{k=0}^{\infty}(2k+1)^{-z} + 2^{-z}\cdot\zeta(z),$$

we obtain

$$\sum_{k=0}^{\infty}(2k+1)^{-z} = \big(1 - 2^{-z}\big)\cdot\zeta(z). \tag{1}$$

Hence, for $\mathrm{Re}(z) > p$, we deduce that

$$\mathrm{tr}\big(|D_\mathcal{G}|^{-z}\big) = 2\sum_{e\in\mathcal{E}}\sum_{k\in\mathbb{N}_0}\left(\frac{2\ell(e)}{(2k+1)\pi}\right)^z$$

$$= \frac{2^{z+1}}{\pi^z}\cdot\left(\sum_{e\in\mathcal{E}}\ell(e)^z\right)\cdot\left(\sum_{k=0}^{\infty}(2k+1)^{-z}\right)$$

$$= \frac{2^{z+1}}{\pi^z}\cdot\big(1 - 2^{-z}\big)\cdot\left(\sum_{e\in\mathcal{E}}\ell(e)^z\right)\cdot\zeta(z).$$

Example 7.13. Let $\mathcal{G}$ be the Cayley graph of $\mathbb{F}_2$. Then, the zeta function of $\mathcal{G}$ has a meromorphic extension to all of $\mathbb{C}$ given by

$$\zeta_{\mathcal{G}}(z) = \frac{8}{\pi^z} \frac{1 - 2^{-z}}{1 - 3 \cdot 2^{-z}} \cdot \zeta(z), \quad \text{for } z \in \mathbb{C}.$$

Indeed, this is true for $\text{Re}(z) > \log 3 / \log 2$, by the last displayed equation in Example 7.11. Hence, by analytic continuation, it is true for all $z \in \mathbb{C}$.

Aside from a trivial multiplicative factor $f(z)$, which is an entire function, the zeta function of $\mathcal{G}$ is of the same form as the spectral zeta function of a self-similar fractal string, which is always equal to the product of the Riemann zeta function and the geometric zeta function of the fractal string (see [26,27,31,32] and Chapter 1 in [29] or [30]).

We refer to [30, Chapters 1, 4 and 5], for the precise definition of the complex dimensions of a fractal string, using the notions of "screen and window."

Definition 7.14. Assume that $\zeta_{\mathcal{G}}$ admits a meromorphic continuation to an open neighborhood of a window $W \subset \mathbb{C}$. Then the *visible complex dimensions* of $\mathcal{G}$ (relative to W) are the poles in W of the meromorphic continuation of $\zeta_{\mathcal{G}}$. The resulting set of visible complex dimension is denoted by $\mathfrak{D}_{\mathcal{G}}(W)$:

$$\mathfrak{D}_{\mathcal{G}}(W) := \{z \in W \mid \zeta_{\mathcal{G}} \text{ has a pole at } z\}.$$

If $W = \mathbb{C}$, then we simply write $\mathfrak{D}_{\mathcal{G}}$ for $\mathfrak{D}_{\mathcal{G}}(\mathbb{C})$ and call the elements of $\mathfrak{D}_{\mathcal{G}}$ the *complex dimensions* of $\mathcal{G}$.

It follows that the set of complex dimensions of $\mathcal{G}$, the Cayley graph of $\mathbb{F}_2$, is given by

$$\mathfrak{D}_{\mathcal{G}} = \{1\} \cup \{\mathcal{D}_{\mathcal{G}} + \sqrt{-1} \cdot k \cdot \mathbf{p} \mid k \in \mathbb{Z}\},$$

where, in the terminology of [29,30], $\mathcal{D}_{\mathcal{G}} := \log 3 / \log 2$ is the Minkowski dimension of $\mathcal{G}$ and $\mathbf{p} := 2\pi / \log 2$ is its oscillatory period.

Note that this is analogous both to what happens for self-similar strings (see [30, Section 2.3.1 and especially Chapter 3], where such strings are allowed to have dimension greater than 1), as was mentioned earlier, and for the Sierpinski drum (see [30, Section 6.6.1]), viewed as a fractal spray (more specifically, as the bounded, infinitely connected planar domain with boundary the Sierpinski gasket). Indeed, it follows from [30, Eqs. (6.81) and (6.82)] that the set of (spectral) complex dimensions of the Sierpinski drum is equal to $\{2\} \cup \mathfrak{D}_{\mathcal{G}}$, with $\mathfrak{D}_{\mathcal{G}}$ as in the last displayed equation. In our present situation, the value 1 appears naturally since it corresponds to the dimension of any edge of the tree, while in [30], the additional value 2 occurs because the Sierpinski drum is viewed as embedded in $\mathbb{R}^2$.

Remark 7.15. More generally, choose a suitable window W contained in the half-plane $\text{Re}(z) > 0$. We deduce from the discussion following Definition 7.12 that the zeta function $\zeta_{\mathcal{G}}(z)$ of any p-summable tree $\mathcal{G}$ (as in Theorem 7.10) is given by

$$\zeta_{\mathcal{G}}(z) := g(z) \cdot \zeta_{\mathcal{L}}(z) \cdot \zeta(z) = g(z) \cdot \zeta_\nu(z).$$

Here, $g(z)$ is an entire function which is nowhere vanishing in W and $\zeta_{\mathcal{L}}(z)$ is the meromorphic continuation to W of the geometric zeta function $\sum_{e \in \mathcal{E}} \ell(e)^z$ of the fractal string $\mathcal{L} = \mathcal{L}_{\mathcal{G}} := \{\ell_e\}_{e \in \mathcal{E}}$ associated with $\mathcal{G}$. Furthermore, $\zeta_\nu(z) := \zeta_{\mathcal{L}}(z) \cdot \zeta(z)$ is the spectral zeta function of $\mathcal{L}$ ([27,32], and [30, Theorem 1.19]). In particular, we have

$$\mathfrak{D}_{\mathcal{G}}(W) = \{1\} \cup \mathfrak{D}_{\mathcal{L}}(W),$$

where $\mathfrak{D}_{\mathcal{L}}(W)$ is the set of visible complex dimension of $\mathcal{L}$ (as in Definition 7.14).

8. The Sierpinski gasket: Hausdorff measure and geodesic metric

The Sierpinski gasket is well known and is described in many places. In particular, it is a connected fractal subset of the Euclidean plane $\mathbb{R}^2$; in fact, it can be viewed as a continuous planar curve which is nowhere differentiable. We refer the reader to the books by Barlow, Edgar, and Falconer [1,12,13,15], which all contain good descriptions and a lot of information about this fractal set. The gasket can be obtained in many ways. The most common is probably the one, illustrated in Fig. 4, where one starts with a solid equilateral triangle in the plane and cut out one open equilateral triangle of half size, and then continue to cut out open triangles of smaller and smaller sizes. Hence, in the nth step, one cuts away 3^{n-1} open equilateral triangles with side length equal to 2^{-n} that of the original one.

In our present study, we would rather like to consider a construction where the Sierpinski gasket, denoted from now on by $\mathcal{SG}$, is obtained as the closure of the limit of an increasing sequence of sets in $\mathbb{R}^2$. This means that we take as a starting point an equilateral triangle, but not solid anymore; just its border, consisting of the 3 sides, and the three vertices. We call this figure $\mathcal{SG}_0$. The next figure, $\mathcal{SG}_1$, is obtained by adding another triangle of half size, and turned upside down relative to $\mathcal{SG}_0$, and so on. This procedure is well known, and illustrated in Fig. 1 in the introduction. We are not so much interested in the algorithm used to construct the Sierpinski gasket. Instead, our goal is to describe the topological properties of the gasket via noncommutative methods. From this point of view, it seems better to describe $\mathcal{SG}_1$ as consisting of 3 equilateral triangles of half size *and glued together at* 3 *points*. Clearly, the set $\mathcal{SG}_0$ is still in a natural way a subset of $\mathcal{SG}_1$. The following figure $\mathcal{SG}_2$ then consists of 3^2 triangles of size 2^{-2} of $\mathcal{SG}_0$, and they are glued together at $3 + 3^2$ points. And finally, $\mathcal{SG}_n$ consists of 3^n triangles, each shrunken to a size 2^{-n} of the starting one, and glued together at $3(3^n - 1)/2$ points. Figures 1 and 4 illustrate, when compared, the well-known fact that the closure of the union of the sets $\mathcal{SG}_n$ equals the intersection of the decreasing sequence of the sets, which are obtained via the cutting procedure.

There are, of course, identifications between points in the figures such that each figure $\mathcal{SG}_n$ is a subset of the next one $\mathcal{SG}_{n+1}$, but this will be taken care of by the construction of a spectral triple for $C(\mathcal{SG})$ as a direct sum of triples associated to each of the triangles which appear in any
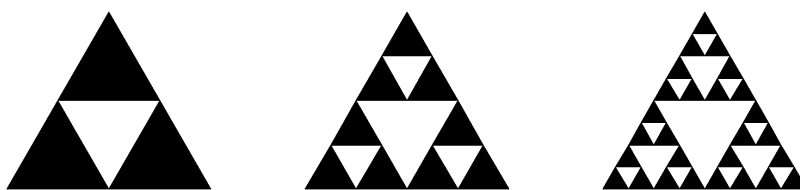

Fig. 4. The first 3 steps in the standard construction of the Sierpinski gasket.

of the figures $\mathcal{SG}_n$. The advantage of this way of constructing a spectral triple for the Sierpinski gasket is that it keeps track of the *holes* in the gasket.

We will use a circle as the basis for the construction of an unbounded Fredholm module for each of the triangles contained in $\mathcal{SG}_n$, and then ultimately obtain a spectral triple for the Sierpinski gasket. In order to avoid using too many π's, we fix the side length of $\mathcal{SG}_0$ to $2\pi/3$. The perimeter of each triangle which is used to form $\mathcal{SG}_n$ is then $2\pi/2^n$. We will think of each of the triangles shown above as having a horizontal edge. Further, we will introduce a natural parameterization of each triangle in $\mathcal{SG}_n$ by using the right-hand corner on the bottom line as the starting point and then run counterclockwise by arclength. In this way, we get for each such triangle in $\mathcal{SG}_n$ an isometry of the circle of radius 2^{-n} onto this triangle, when both are equipped with arclength as metric. We will now introduce a numbering of the triangles which go into this construction and also define the associated spectral triples.

Definition 8.1.

 (i) Given n in $\mathbb{N}_0$, choose a numbering of the 3^n triangles of size 2^{-n} which form $\mathcal{SG}_n$, and let $\Delta_{n,i}, i \in \{1, 2, \ldots, 3^n\}$, denote the numbered triangles.
(ii) Let, for each n in $\mathbb{N}_0$ and each i in $\{1, \ldots, 3^n\}$, the mapping $r_{n,i} : [-2^{-n}\pi, 2^{-n}\pi] \to \Delta_{n,i}$ be defined such that $r_{n,i}(0)$ equals the lower right-hand corner of $\Delta_{n,i}$ and the mapping is an isometry, modulo $2^{1-n}\pi$, of this interval onto the triangle $\Delta_{n,i}$ equipped with the geodesic distance as metric, and the counterclockwise orientation. The mapping $r_{n,i}$ induces a surjective homomorphism $\Phi_{n,i}$ of $C(\mathcal{SG})$ onto $C([-2^{-n}\pi, 2^{-n}\pi])$ by

$$\forall t \in \left[-2^{-n}\pi, 2^{-n}\pi\right], \forall f \in C(\mathcal{SG}): \quad \Phi_{n,i}(f)(t) := f\big(r_{n,i}(t)\big).$$

(iii) Let, for each n in $\mathbb{N}_0$ and each i in $\{1, \ldots, 3^n\}$, the unbounded Fredholm module $ST_{n,i}(\mathcal{SG}) := (C(\mathcal{SG}), H_{n,i}, D_{n,i})$ for $\mathcal{SG}$ be given by
   (1) $H_{n,i} := H_{2^{-n}}$ (see Definition 2.1);
   (2) the representation $\pi_{n,i} : C(\mathcal{SG}) \to B(H_{n,i})$ is defined for f in $C(\mathcal{SG})$ as the multiplication operator which multiplies by the function $\Phi_{n,i}(f)$.
   (3) $D_{n,i} := D_{2^{-n}}^t$ (Definition 2.2).

Theorem 8.2. *The direct sum of all the unbounded Fredholm modules $ST_{n,i}(\mathcal{SG})$ for n in $\mathbb{N}_0$ and i in $1, \ldots, 3^n$ gives a spectral triple for $\mathcal{SG}$. This spectral triple is denoted $ST(\mathcal{SG}) = (C(\mathcal{SG}), H_{\mathcal{SG}}, D_{\mathcal{SG}})$, and it is s-summable if and only if $s > \log 3/\log 2$. Hence, its metric dimension is equal to $\log 3/\log 2$, which is also the Minkowski (as well as the Hausdorff) dimension of the Sierpinski gasket.*

Proof. As usual, we have to prove that the algebra of continuous functions on $\mathcal{SG}$, which have bounded commutators with the Dirac operator $D_{\mathcal{SG}}$, is dense in $C(\mathcal{SG})$. As before, it is enough to show that this algebra separates the points in $\mathcal{SG}$. In this case, our task is easy. We just remark that the real-valued linear functions on $\mathbb{R}^2$ separate points and we will prove that any real-valued linear functional on $\mathbb{R}^2$, say $\psi(x, y) = ax + by$, has the property that its restriction to $\mathcal{SG}$ has a bounded commutator with the sum of the Dirac operators. We will think of each of the triangles shown above as having an horizontal edge, as before. Moreover, we will consider a Euclidean coordinate system such that the x-axis is also horizontal. Let us then consider the composition of ψ with any of the parameterization mappings $r_{n,i}$. This yields a continuous $2^{1-n}\pi$-periodic

function, say f, on the interval $[-2^{-n}\pi, 2^{-n}\pi]$ or rather on $\mathbb{R}$, which is affine on the intervals $[-2^{1-n}\pi/3, 0]$, $[0, 2^{1-n}\pi/3]$, $[-2^{-n}\pi, -2^{1-n}\pi/3]$, and $[2^{1-n}\pi/3, 2^{-n}\pi]$. The slopes of all the functions $\psi \circ r_{n,i}$ will belong to the set

$$S := \left\{ a, \frac{-a+b\sqrt{3}}{2}, \frac{-a-b\sqrt{3}}{2} \right\}.$$

Note that S is independent of both n and i. According to Lemma 2.3, for each pair n, i the function $\psi \circ r_{n,i}$ is in the domain of $D_{n,i}$ and $\|[\pi_{n,i}(f), D_{n,i}]\| \leqslant \max(S)$. Hence, $\|[\pi_{\mathcal{SG}}(f), D_{\mathcal{SG}}]\| \leqslant \max(S)$ and we have proven that the restriction of any affine function f on $\mathbb{R}^2$ to $\mathcal{SG}$ has the property that $\|[\pi_{\mathcal{SG}}(f), D_{\mathcal{SG}}]\|$ is bounded.

In order to check that $(D_{\mathcal{SG}}^2 + I)^{-1}$ is compact and establish its summability properties, we compute for each $n \in \mathbb{N}_0$ the set of eigenvalues E_n of $D_{\mathcal{SG}}$, which are added on in the nth step.

$n = 0$: $E_0 = \{(2k+1)/2 \mid k \in \mathbb{Z}\}$, each of multiplicity $1 = 3^0$.
$n > 0$: $E_n = \{2^{n-1}(2k+1) \mid k \in \mathbb{Z}\}$, each of multiplicity 3^n.

(Note that the formula for $E_n (n > 0)$ is valid for $n = 0$ as well. However, by scaling, it is naturally derived from that for E_0.)

Now that we know the eigenvalues, we can compute the trace of the operator $|D_{\mathcal{SG}}|^{-z}$ for a given complex number z, and we will in the next theorem show that the trace is finite for $\mathrm{Re}(z) > \log 3 / \log 2$. $\square$

Remark 8.3. It is well known that the Sierpinski gasket has Minkowski and Hausdorff dimensions equal to $\log 3 / \log 2$. Indeed, it follows from the fact that $\mathcal{SG}$ is a self-similar set satisfying the Open Set Condition and can be constructed out of 3 similarity transformations each of scaling ratio $1/2$. (See e.g. [15, Chapter 9].)

Theorem 8.4. *Let $\zeta(z)$ denote the Riemann zeta function. Then, for any complex number z such that* $\mathrm{Re}(z) > \log 3 / \log 2$,

$$\mathrm{tr}\big(|D_{\mathcal{SG}}|^{-z}\big) = 2^{z+1} \cdot \frac{1 - 2^{-z}}{1 - 3 \cdot 2^{-z}} \cdot \zeta(z).$$

Moreover, for any Dixmier trace Tr_ω on $B(H_{\mathcal{SG}})$, we have

$$\mathrm{Tr}_\omega\big(|D_{\mathcal{SG}}|^{-\frac{\log 3}{\log 2}}\big) = \frac{4}{\log 3} \cdot \zeta\left(\frac{\log 3}{\log 2}\right).$$

Proof. We recall here the formula (1) from page 65:

$$\sum_{k=0}^{\infty} (2k+1)^{-z} = \big(1 - 2^{-z}\big) \cdot \zeta(z).$$

Hence, for $\mathrm{Re}(z) > \log 3 / \log 2$ we deduce that

$$\mathrm{tr}\big(|D_{\mathcal{SG}}|^{-z}\big) = \sum_{n\in\mathbb{N}_0}\sum_{k\in\mathbb{Z}} 3^n 2^{-nz}|k+1/2|^{-z}$$

$$= \sum_{n\in\mathbb{N}_0} 3^n 2^{-nz} \sum_{k\in\mathbb{Z}} 2^z |2k+1|^{-z}$$

$$= \frac{2^z}{1-3\cdot 2^{-z}} \cdot 2 \cdot \sum_{k=0}^{\infty} |2k+1|^{-z}$$

$$= 2^{z+1}\cdot \frac{1-2^{-z}}{1-3\cdot 2^{-z}}\cdot \zeta(z).$$

Then, by Proposition 4 on page 306 in [6], we obtain

$$\mathrm{Tr}_\omega\big(|D_{\mathcal{SG}}|^{-\frac{\log 3}{\log 2}}\big) = \lim_{x\to 1+}(x-1)\,\mathrm{tr}\big((|D_{\mathcal{SG}}|^{-\frac{\log 3}{\log 2}})^x\big)$$

$$= \lim_{x\to 1+}(x-1)\cdot 2^{x\frac{\log 3}{\log 2}+1}\cdot \frac{1-2^{-x\frac{\log 3}{\log 2}}}{1-3\cdot 2^{-x\frac{\log 3}{\log 2}}}\cdot \zeta\Big(x\frac{\log 3}{\log 2}\Big)$$

$$= \lim_{x\to 1+}\frac{x-1}{1-3^{1-x}}2\big(3^x-1\big)\cdot \zeta\Big(x\frac{\log 3}{\log 2}\Big)$$

$$= \frac{4}{\log 3}\cdot \zeta\Big(\frac{\log 3}{\log 2}\Big).$$

We note that it is also possible to compute the classical limit expression for the Dixmier trace of a measurable operator, obtained as the limit as $N\to\infty$ of $(1/\log N)\sum_{j=0}^{N-1}|\lambda_j|^{-(\log 3/\log 2)}$, where $(\lambda_j)_{j=1}^{\infty}$ are the characteristic values (here, the eigenvalues) of the given compact operator, written in nonincreasing order. $\square$

Remark 8.5. The zeta function of the Sierpinski gasket can now be defined for $\mathrm{Re}(z) > \log 3/\log 2$ by $\zeta_{SG}(z) := \mathrm{tr}(|D_{SG}|^{-z})$. In view of Theorem 8.4, it has a meromorphic continuation to all of $\mathbb{C}$ and is given by

$$\zeta_{SG}(z) = 2^{z+1}\cdot \frac{1-2^{-z}}{1-3\cdot 2^{-z}}\cdot \zeta(z), \quad \text{for } z\in\mathbb{C}.$$

It also follows from the foregoing formula that the zeta function of the Sierpinski gasket (as defined just above) and the zeta function for the Cayley tree given in Example 7.13 are proportional modulo the function $h(z) = \pi^z\cdot 2^{z-2}$. Hence, they have the same set of complex dimensions (see Section 7.1), given by

$$\mathfrak{D}_{SG} = \{1\}\cup\big\{\mathcal{D}_{SG}+\sqrt{-1}\cdot k\cdot \mathbf{p}\,\big|\,k\in\mathbb{Z}\big\},$$

where $\mathcal{D}_{SG} := \log 3/\log 2$ is the Minkowski dimension of the gasket and $\mathbf{p} := 2\pi/\log 2$ is its oscillatory period. This is in agreement with a conjecture made in [28, Section 8], when discussing the 'geometric complex dimensions' of the gasket. We note, however, that the value 1 was not included in [28] but appears naturally in our context since it corresponds to the dimension of the

boundary of any of the holes (circles or triangles) of the gasket. Following [28], we may refer to $\mathfrak{D}_{SG}$ as the set of *geometric complex dimensions* of the Sierpinski gasket.

A Dixmier trace, Tr_ω, on $B(H_{SG})$ induces a positive linear functional, τ, on $\mathrm{C}(SG)$, as stated in [6, Proposition 5, Chapter IV.2], and proved in [4]. It turns out that τ is a nonzero multiple of the integral with respect to the Hausdorff measure, say μ, on the Sierpinski gasket (see e.g. [20,21], or [39]), which, in turn, has a very natural description in terms of functional analytic concepts. To show this, we will first give a description of the Hausdorff integral, which can be used to establish the proportionality between τ and the Hausdorff measure (or rather, integral). Having this to our disposal, we can deduce from the second part of Theorem 8.4 that for any continuous function f on the gasket, we have

$$\tau(f) := \mathrm{Tr}_\omega\left(\pi_{SG}(f)|D_{SG}|^{-\frac{\log 3}{\log 2}}\right) = \frac{4}{\log 3} \cdot \zeta\left(\frac{\log 3}{\log 2}\right) \cdot \int_{SG} f(x)\,d\mu(x).$$

In order to establish this relation, we will use the description of the gasket as an increasing sequence SG_n of graphs, but this time we will, for a given nonnegative integer n, focus on the set consisting of the 3^{n+1} midpoints of the sides in the 3^n triangles $\Delta_{n,i}$ of size $2^{1-n}\pi/3$ contained in SG_n. We refer the reader to Fig. 1 in the introduction for the pictures of the first few sets SG_n and now begin the formal description.

The triangles in SG_n are denoted $\{\Delta_{n,i} \mid 1 \leqslant i \leqslant 3^n\}$ and we denote the midpoints of a triangle $\Delta_{n,i}$ by $\{x_{n,i,j} \mid 1 \leqslant j \leqslant 3\}$. For any natural number n, we then define a positive linear functional ψ_n of norm 1, a state, on $\mathrm{C}(SG)$ by

$$\forall f \in \mathrm{C}(SG): \quad \psi_n(f) := 3^{-(n+1)} \sum_{i=1}^{3^n} \sum_{j=1}^{3} f(x_{n,i,j}).$$

Proposition 8.6. *Let μ denote the Hausdorff probability measure on the Sierpinski gasket SG and let ψ denote the state on $\mathrm{C}(SG)$ defined by*

$$\forall f \in \mathrm{C}(SG): \quad \psi(f) := \int_{SG} f(t)\,d\mu(t).$$

Then the sequence of states $(\psi_n)_{n\in\mathbb{N}}$ converges to ψ in the weak*-topology on the dual of $\mathrm{C}(SG)$.

Proof. We will first show that for each complex continuous function f on SG, the sequence $(\psi_n(f))_n$ is a Cauchy sequence. This follows from the fact that any such function f is uniformly continuous and the points $x_{n,i,j}$ are evenly distributed on the gasket. To be more precise, let $\varepsilon > 0$ be given, then choose, by the uniform continuity of f, an n_0 in $\mathbb{N}$ so large that for each h in $\{1, \ldots, 3^{n_0}\}$ and any two points x, y in SG which are inside or on the triangle $\Delta_{n_0,h}$, we have $|f(x) - f(y)| \leqslant \varepsilon$. Let then n be a natural number bigger than n_0 and let us consider the average, say $\nu_{n_0,h}^n(f)$, of the values $f(x_{n,i,j})$ over all the midpoints $x_{n,i,j}$ from SG_n which are on the border or inside the triangle $\Delta_{n_0,h}$. Then the estimate for the variation of f over the points inside or on $\Delta_{n_0,h}$ implies that

$$\left| v_{n_0,h}^n(f) - \frac{f(x_{n_0,h,1}) + f(x_{n_0,h,2}) + f(x_{n_0,h,3})}{3} \right| \leqslant \varepsilon.$$

We can then establish the Cauchy property by using the following inequalities:

$$\forall n \geqslant n_0: \quad \left| \psi_n(f) - \psi_{n_0}(f) \right| = \left| 3^{-n_0} \sum_{h=1}^{3^{n_0}} \left(v_{n_0,h}^n(f) - \frac{f(x_{n_0,h,1}) + f(x_{n_0,h,2}) + f(x_{n_0,h,3})}{3} \right) \right|$$
$$\leqslant \varepsilon.$$

This shows that the sequence $(\psi_n)_{n\in\mathbb{N}}$ is weak*-convergent to a state, say ϕ, on $C(\mathcal{SG})$. By construction, it is clear that ϕ is self-similar with respect to the basic affine contractions which define Sierpinski's gasket, as described for instance by Strichartz in [39]. Hence, $\phi = \psi$ and the proposition follows. $\square$

Having this to our disposal, we can establish the claimed relationship between τ and ψ.

Theorem 8.7. *Let τ be the functional on $C(\mathcal{SG})$ given by*

$$\tau(f) := \mathrm{Tr}_\omega\big(\pi_{\mathcal{SG}}(f)|D_{\mathcal{SG}}|^{-\frac{\log 3}{\log 2}}\big)$$

and μ the Hausdorff probability measure on $\mathcal{SG}$. Then, for any continuous complex-valued function f on $\mathcal{SG}$, we have

$$\tau(f) = \frac{4}{\log 3} \cdot \zeta\left(\frac{\log 3}{\log 2}\right) \cdot \int_{\mathcal{SG}} f(x)\,d\mu(x) = \frac{4}{\log 3} \cdot \zeta\left(\frac{\log 3}{\log 2}\right) \cdot \psi(f).$$

Proof. Let f be a continuous *real-valued* function on $\mathcal{SG}$ and $\varepsilon > 0$ a positive real number. Let us then go back to the proof of Proposition 8.6 and choose $n_0 \in \mathbb{N}$ and, for $n > n_0$ and $h \in \{1, \dots, 3^{n_0}\}$, define $v_{n_0,h}^n(f)$ as above. We then restrict our attention to the portion of the Sierpinski gasket which is contained inside or on the triangle $\Delta_{n_0,h}$ and denote this space by $\mathcal{SG}_{n_0,h}$. It follows that for the function which is constant 1 on $\mathcal{SG}_{n_0,h}$, say $I_{n_0,h}$, and for $f_{n_0,h}$ the analogous restriction of f, we have in the natural ordering on $C(\mathcal{SG}_{n_0,h})$,

$$\big(v_{n_0,h}^n(f) - \varepsilon\big) I_{n_0,h} \leqslant f_{n_0,h} \leqslant \big(v_{n_0,h}^n(f) + \varepsilon\big) I_{n_0,h}.$$

For each $h \in \{1, \dots, 3^{n_0}\}$, we can naturally define a spectral triple for $\mathcal{SG}_{n_0,h}$ by deleting all the summands of $ST(\mathcal{SG})$ which are based on triangles outside $\Delta_{n_0,h}$. To any such triple, we can associate a corresponding functional $\tau_{n_0,h}$ and we get

$$\tau(f) = \sum_{h=1}^{3^{n_0}} \tau_{n_0,h}(f_{n_0,h}) \quad \text{and} \quad \tau_{n_0,h}(I_{n_0,h}) = 3^{-n_0} \tau(I).$$

Since τ is a positive functional, the inequalities above give

$$\sum_{h=1}^{3^{n_0}}\left(v_{n_0,h}^n(f)-\varepsilon\right)\left(3^{-n_0}\tau(I)\right) \leqslant \tau(f) \leqslant \sum_{h=1}^{3^{n_0}}\left(v_{n_0,h}^n(f)+\varepsilon\right)\left(3^{-n_0}\tau(I)\right),$$

where I is the identity for the algebra of continuous functions on $\mathcal{SG}$. If we then go back to the proof of Proposition 8.6, we see that

$$\left|\psi(f)-3^{-n_0}\sum_{h=1}^{3^{n_0}}v_{n_0,h}^n(f)\right| \leqslant \varepsilon, \quad \text{so} \quad \left|\tau(I)\psi(f)-\tau(f)\right| \leqslant \tau(I)\varepsilon.$$

By the second part of Theorem 8.4, we know that

$$\tau(I) = \frac{4}{\log 3} \cdot \zeta\left(\frac{\log 3}{\log 2}\right),$$

and the theorem follows. $\square$

Remark 8.8. In [17], Guido and Isola have associated a spectral triple to a general self-similar fractal in $\mathbb{R}$. For such a spectral triple, they proved the same type of result as the one stated in Theorem 8.7. It is possible that their proof of [17, Lemma 4.10], can be adapted to our present case, from which Theorem 8.7 would then follow by using the uniqueness of a normalized self-similar measure on the gasket having the same homogeneity as the $(\log 3/\log 2)$-Hausdorff measure. However, since the functional ψ of Theorem 8.6 offers a nice description of the Hausdorff measure on the Sierpinski gasket, we have preferred to give a direct argument which shows that the positive functionals τ and ψ on $C(\mathcal{SG})$ are proportional.

Remark 8.9. In the more delicate context of standard analysis on fractals, Kigami and Lapidus identify in [22] the volume measure constructed in [27] via a Dixmier trace functional. In particular, they show that this measure is self-similar but is not always proportional to the natural Hausdorff measure on the self-similar fractal, even when the measure maximizes the spectral exponent (i.e., is an analogue of 'Riemannian volume,' in the sense of [27,28]). In the latter case, however, and for the special case of the standard Sierpinski gasket, it does coincide with the natural Hausdorff measure, pointing to some possible connections or analogies between the two points of view.

We will now discuss the concept of geodesic distance on $\mathcal{SG}$ and then show that it can actually be measured by the metric induced by the spectral triple $ST(\mathcal{SG})$. The geodesic distance between two points p and q on the gasket is denoted $d_{\text{geo}}(p,q)$, and it is defined as the minimal length of a rectifiable continuous curve on the gasket connecting p and q. This metric on the gasket is studied in Barlow's lecture notes [1] and from page 13 of these notes we quote the proposition below, which shows that the geodesic metric on the gasket is Lipschitz equivalent to the restriction of the Euclidean metric. In particular, the geodesic metric induces the standard topology on the gasket.

Proposition 8.10. *Let p, q be arbitrary points in $\mathcal{SG}$ and let $\|p-q\|$ denote their Euclidean distance. Then*

$$\|p-q\| \leqslant d_{\text{geo}}(p,q) \leqslant 8\|p-q\|.$$

We have not found an easy way to give an analytic expression of the geodesic distance between two arbitrary points in the gasket, but the distance from a corner point, a vertex, in $\mathcal{SG}_0$ to an arbitrary point in $\mathcal{SG}$ can be precisely expressed in terms of barycentric coordinates, as we now show.

Lemma 8.11. *Let v_1, v_2, v_3 be the vertices of $\mathcal{SG}_0$. For any point p in $\mathcal{SG}$, let (x, y, z) denote the barycentric coordinates of p with respect to v_1, v_2, v_3. Then the geodesic distance in $\mathcal{SG}$ from v_1 to p is $y + z$.*

Proof. Since the Euclidean metric is equivalent to the geodesic metric, and the function which assigns barycentric coordinates to a point is continuous with respect to these metrics, we see that it is enough to prove the statement for a point p which is a vertex in one of the triangles from one of the sets $\mathcal{SG}_n$ where n is in $\mathbb{N}$. Let then n in $\mathbb{N}$ be fixed and let p denote a vertex in $\mathcal{SG}_n$, and let us consider paths in $\mathcal{SG}_n$ which connect v_1 to p; we will then show that a geodesic curve can be obtained inside $\mathcal{SG}_n$. Any path from v_1 to p in $\mathcal{SG}_n$ will be a sum of steps, each of which will be a positive multiple of one of the following 6 vectors, which all have length $2\pi/3$.

$$v_2 - v_1, \; v_1 - v_2, \; v_3 - v_1, \; v_1 - v_3, \; v_3 - v_2, \; v_2 - v_3.$$

When $p = (x, y, z)$ in barycentric coordinates, we have $p = v_1 + y(v_2 - v_1) + z(v_3 - v_1)$; so it must follow that the geodesic distance between v_1 and p is at least $y + z$. On the other hand, it is possible, via a little drawing, to see that $\mathcal{SG}_n$ contains a path of length $y + z$ between v_1 and p. The geodesic distance between v_1 and p is then $y + z$, and this result extends to a general point p in $\mathcal{SG}$, by continuity. □

Let us then consider the geodesic distance between two arbitrary, but different, points p and q. In order to describe a way to compute this distance, we will look at the other picture of $\mathcal{SG}$, as the limit of a decreasing sequence, say $\mathcal{F}_n$, of compact subsets of the largest solid triangle, as depicted in Fig. 4. For any nonnegative integer n, $\mathcal{F}_n$ is the union of 3^n equilateral solid triangles, say $\{T_{n,k} \mid 1 \leqslant k \leqslant 3^n\}$, with side length $2^{-n}2\pi/3$. In order to determine the geodesic distance between p and q, we determine the largest number $n_0 \in \mathbb{N}_0$ for which there exists a k_0 in $\{1, 2, \ldots, 3^{n_0}\}$ such that both p and q belong to T_{n_0,k_0}. From the solid triangle T_{n_0,k_0} remain 3 solid triangles, say T_{n_0+1,k_1}, T_{n_0+1,k_2}, T_{n_0+1,k_3}, in the next step of the iterative construction. By assumption, p and q must lie in two different of these smaller triangles, say T_{n_0+1,k_1} and T_{n_0+1,k_2}. Hence, any path from p to q must run from p to a vertex in T_{n_0+1,k_1}. We can then use Lemma 8.11 to measure the length of this part of the path. The vertex we have arrived at may, or may not, be in T_{n_0+1,k_2} too. In the first case, we can measure the distance from the corner to q, again using Lemma 8.11. In the second case, the corner of T_{n_0+1,k_1} at which we have arrived must be directly connected to a corner of T_{n_0+1,k_2} via a side of the third triangle T_{n_0+1,k_3}. This side has length $2^{-(n_0+1)}2\pi/3$, and this edge will bring us to a corner in T_{n_0+1,k_2}, from where we can measure the distance to q on the basis of Lemma 8.11. This observation has several consequences, some of which we will formulate in the next results.

Lemma 8.12. *Let q be a point in $\mathcal{SG}$ and let g be the continuous function on $\mathcal{SG}$ defined by $g(p) = d_{\mathrm{geo}}(p, q)$. Then, for any continuous curve $r_{n,i}$ which parameterizes a triangle in the gasket, we have $\|[D_{r_{n,i}}, \pi_{r_{n,i}}(g)]\| \leqslant 1$.*

Proof. Let us return to the introductory example in Lemma 8.11 and suppose for simplicity that q is the vertex v_1. Given an edge, say e, in a triangle $\Delta_{n,i}$, which is parameterized by one of the functions $r_{n,i}$, then there are only 3 possibilities for the slope of the edge, since it must be parallel to one of the edges of the big triangle. Following Lemma 8.11, we therefore deduce that in this case where $q = v_1$, we must have

$$\exists \beta \in \mathbb{R},\ \exists \alpha \in \{-1, 0, 1\}: \quad r_{n,i}(t) \in e \quad \Rightarrow \quad g\big(r_{n,i}(t)\big) = d_{\mathrm{geo}}\big(r_{n,i}(t), v_1\big) = \alpha t + \beta.$$

According to Lemma 2.3, such a function is in the domain of all the operators $D_{n,i}$, and we see that the derivative is numerically bounded by 1. We will now establish a similar result for a general point q. This situation is discussed in the text just in front of this lemma and it follows from there that the function $g(r_{n,i}(t))$ has to be modified by an additive constant, which measures the geodesic distance from q to one of the endpoints of the edge e. When $r_{n,i}(t)$ passes along the edge e, the geodesic distance from q may reach an extremal value in the interior of the edge, and the slope may change, but still be in the set $\{-1, 0, 1\}$. Hence, according to Lemma 2.3, we deduce that the derivative of $g(r_{n,i}(t))$ exists almost everywhere and is numerically bounded by 1, and by the same lemma we get $\|[D_{r_{n,i}}, \pi_{r_{n,i}}(g)]\| \leqslant 1$. $\quad\square$

Theorem 8.13. *The metric on the Sierpinski gasket, $d_{\mathcal{SG}}$, induced by the spectral triple* $(C(\mathcal{SG}), H_{\mathcal{SG}}, D_{\mathcal{SG}})$, *coincides with the geodesic distance on* $\mathcal{SG}$.

Proof. As before, let $\mathcal{N} = \{f \in C(\mathcal{SG}) \mid \|[D_{\mathcal{SG}}, \pi_{\mathcal{SG}}(f)]\| \leqslant 1\}$. Then Lemma 8.12 shows that all the functions of the form $g(p) = d_{\mathrm{geo}}(p, q)$ belong to $\mathcal{N}$. We therefore have

$$\forall p, q \in \mathcal{SG}: \quad d_{\mathrm{geo}}(p, q) \leqslant d_{\mathcal{SG}}(p, q).$$

The other inequality is an application of the fundamental theorem of calculus. Let points p, q in $\mathcal{SG}$ be given and let f be in $\mathcal{N}$. Then f is continuous and for any $\varepsilon > 0$ there exists a natural number n and vertices v, w in $\mathcal{SG}_n$ such that

$$d_{\mathrm{geo}}(p, v) + d_{\mathrm{geo}}(q, w) + \big|f(p) - f(v)\big| + \big|f(q) - f(w)\big| < \varepsilon. \tag{2}$$

Let us then look at $|f(v) - f(w)|$ and show that this quantity is at most $d_{\mathrm{geo}}(v, w)$. We will therefore consider a path along some of the edges of some of the triangles $\Delta_{n,i}$ connecting v and w inside $\mathcal{SG}_n$. Since f is in $\mathcal{N}$, it follows from Lemma 2.3 that the derivative of the induced function $f(r_{n,i}(t))$ corresponding to an edge in $\Delta_{n,i}$ is a measurable function which is numerically bounded by 1 almost everywhere. We can then express $f(v) - f(w)$ as a sum of line integrals. Each of these integrals is numerically bounded from above by the length of the path over which the integration is performed; so $|f(v) - f(w)|$ is dominated by the length of any path in $\mathcal{SG}_n$ connecting s and t. This implies that

$$\big|f(v) - f(w)\big| \leqslant d_{\mathrm{geo}}(v, w)$$

and, by Eq. (2),

$$\big|f(p) - f(q)\big| \leqslant d_{\mathrm{geo}}(p, q) + \varepsilon.$$

From here we see that

$$d_{\mathcal{SG}}(p,q) = \sup_{f \in \mathcal{N}} \left| f(p) - f(q) \right| \leqslant d_{\mathrm{geo}}(p,q),$$

and the theorem follows. $\quad\square$

9. Concluding remarks

We close this paper by indicating several possible extensions or directions of future research related to this work, some of which may be investigated in later articles:

(i) exploring the possibility of associating to other fractals built on curves spectral triples which are based on a direct sum of r-triples and describing the topological and geometric properties of those fractals;

(ii) investigating the geometric and topological properties of the Sierpinski gasket described by the spectral triples based on some other possible direct sums of spectral triples for circles;

(iii) extending connections and/or relations to other known constructions of differential operators on the Sierpinski gasket and other self-similar fractals, especially in some of the approaches to *analysis on fractals* expounded, for example, in [1] and [20], either probabilistically or analytically. In particular, investigating some of the open problems or conjectures proposed within that framework in [27,28], and aimed at merging aspects of fractal, spectral and noncommutative geometry;

(iv) studying the differential operators (including 'Laplacians') connected to the Dirac-type operators constructed in this paper, as well as of the solutions of partial differential equations naturally associated to them;

(v) looking at the Sierpinski gasket via the 'harmonic coordinates' attached to the Laplacian Δ associated to our Dirac operator D (namely, $-\Delta = D^2$), or to a suitable modification thereof. (See e.g. [40,41] and the relevant references therein for the analogous situation involving the usual Laplacian on the gasket.);

(vi) looking for further applications of the spectral triples as a tool for computing invariants of algebraic topological type for the Sierpinski gasket and other fractals.

We now explain more precisely what we mean by item (ii):

We first recall the construction where the Sierpinski gasket is obtained as the limit of an increasing sequence of sets in $\mathbb{R}^2$ (see Fig. 1). We take as a starting point an equilateral triangle with side length $2\pi/3$ and we call it $\mathcal{SG}_0$. The next figure, $\mathcal{SG}_1$, is obtained by adding another triangle of half size, and turned upside down relative to $\mathcal{SG}_0$, and so on. This means that $\mathcal{SG}_1$ consists of $\mathcal{SG}_0$, and 1 equilateral triangle of size 2^{-1} of $\mathcal{SG}_0$. The following figure $\mathcal{SG}_2$ then consists of $\mathcal{SG}_0$, 1 equilateral triangle of size 2^{-1} of $\mathcal{SG}_0$, and 3 of size 2^{-2} of $\mathcal{SG}_0$. And finally, $\mathcal{SG}_n$ consists of $\mathcal{SG}_0$, 1 equilateral triangle of size 2^{-1} of $\mathcal{SG}_0$, 3 of size 2^{-2} of $\mathcal{SG}_0$, and so on up to 3^{n-1} triangles of size 2^{-n} that of the starting one. We will use a circle as the basis for the construction of a spectral triple for each of the triangles contained in $\mathcal{SG}_n$ as $n \to \infty$, and then ultimately obtain a spectral triple for the Sierpinski gasket in the same way as it was done in Section 8 of the present article. Another possibility is to use a circle as the basis for the construction of a spectral triple for each of the triangles considered in Section 8, as well as each of the triangles considered just above, and then ultimately obtain a spectral triple for the Sierpinski gasket in the same way as it was done in Section 8.

Next, in the special case of the Sierpinski gasket, we elaborate on item (vi) above regarding the computation of topological invariants:

The spectral triple $ST(\mathcal{SG})$ has been constructed in such a way that the theory of [6, Sections IV.1 and IV.2], will yield nontrivial topological information. The details of this will appear in a later article, but to indicate what sort of results we have in mind, we will just describe the bounded Fredholm module (K-cycle) which one can obtain from $ST(\mathcal{SG}) = (C(\mathcal{SG}), H_{\mathcal{SG}}$ (together with $\pi_{\mathcal{SG}}), D_{\mathcal{SG}})$. The unitary part, $F_{\mathcal{SG}}$, of the polar decomposition of the Dirac operator $D_{\mathcal{SG}}$ gives a bounded odd Fredholm module $(\pi_{\mathcal{SG}}, H_{\mathcal{SG}}, F_{\mathcal{SG}})$, see [6, Chapter IV, Section 2] and [38]. The pairing with $K_1(\mathcal{SG})$ can then be obtained in the following way. Let $F_{\mathcal{SG}} = P_+ - (I - P_+)$, where P_+ is the orthogonal projection of $H_{\mathcal{SG}}$ onto the eigenspace corresponding to the positive eigenvalues of $D_{\mathcal{SG}}$, and let f be a continuous function on $\mathcal{SG}$ which has the property that $\pi_{\mathcal{SG}}(f)$ is invertible, i.e. $\min\{|f(x)| \mid x \in \mathcal{SG}\} > 0$. Then the equicontinuity of f will imply that there exists a natural number, say n_0, such that for any $n \geqslant n_0$ and any triangle $\Delta_{n,i}$ of size 2^{-n} that of the original one, the function f has winding number 0 along $\Delta_{n,i}$. This implies that the operator $P_+\pi_{\mathcal{SG}}(f)P_+$ has a finite index which actually is the opposite number to the sum of all the winding numbers over all the triangles $\Delta_{j,h}$ for $0 \leqslant j < n_0$ and $h \in \{1, \ldots, 3^j\}$. This is a rather obvious extension of well-known results for the circle, but we think that also the newer invariants related to the cyclic cohomology of the gasket may be expressed via the spectral triple $ST(\mathcal{SG})$.

Acknowledgments

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