Math10A Midterm solution Winter 2006

I give some solutions, as well as "typical" grading policy. If you happen to be an exceptional case: i.e. you made a "unique" mistake, it is not written here.

1. Let $P = (1, 1, 0)$, $Q = (2, 2, 1)$, $R = (0, 1, 1)$.
(a)[10pt] Find a parametric equation of the plane that contains P, Q, R.
(b) [10pt] Find an equation of the same plane, without parameters (hint: use crossed product).
(c) [5pt] Find the area of the triangle spanned by P, Q, R.

(a): $\vec{v} := \vec{P}Q = (2, 2, 1) - (1, 1, 0) = (1, 1, 1)$, $\vec{w} := \vec{P}R = (0, 1, 1) - (1, 1, 0) = (-1, 0, 1)$. Using these, a parametric equation of the plane is given by $\pi(t, s) = (1, 1, 1)t + (-1, 0, 1)s + (1, 1, 0)$. Most people got it.
(b) First we want a vector which is perp to π, that's given by $\vec{v} \times \vec{w}$. That's $(1, -2, 1)$. Then for any $(x, y, z) = X$ sitting on π, $\vec{X} - \vec{P}$ is perp to it: we get an equation $(\vec{X} - \vec{P}) \cdot \vec{v} \times \vec{w} = 0$, i.e. $(x - 1, y - 1, z) \cdot (1, -2, 1) = (x - 1) - 2(y - 1) + z = 0$. Some of you didn't remember crossed product correctly. But if you proceed finding the equation of the plane with your wrong crossed product, you got 5pts. If you get the crossed product right but didn't know what to do next, you got 5pts as well.
(c) The norm of $\vec{v} \times \vec{w}$ is the area of rectangle spanned by $\vec{v}$ and $\vec{w}$, i.e. twice as large as the triangle spanned by P, Q, R. If you forgot to make it half you lost 1pt.

2.[10] Find the orthogonal projection of $\vec{v} = (2, 1, 0)$ on $\vec{w} = (1, 1, 1)$.

Note that the final answer must be parallel to $\vec{w}$, because you are projecting $\vec{v}$ onto $\vec{w}$. However a lot of you used the formula with v and w switched and didn't even think about your answer. If you did so, you got 5pt. It is important to think about what you are doing geometrically, and think if your final answer makes sense in terms of what you are doing.

3.[10] Compute the determinant of the following matrices.

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 0 & 3 \\ -2 & -1 & 1 \end{pmatrix}.$$

You may do straightforward by definition, or use row deformation. Remember that determinant give a NUMBER, not a vector;;

4.(a)[10] Sketch the level curves of $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2 + y^2$.
(b) [10] Sketch the graph of $z = f(x, y)$ in $\mathbb{R}^3$. Do your best.

(a): you get circles for $c > 0$, a dot at the origin for $c = 0$.
(b): you get a rotated parabola. If you say that it's a parabola by looking from side, putting $x = 0$, get $z = y^2$ and stuff, that's perfect. If you just say parabola, or draw a picture which clearly shows that you intended it to be a parabola but didn't write a mathematical justification, you lost only

1pt. If you don't seem to know that it's parabola, or don't seem to care what it is but still draw something ranging from a bowl to a cocktail glass looking up, you lost 3pt.

5. Find the following limits if it exist. If it doesn't, explain why.
(a)[10] $\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+2y^2}}$
(b) [10] $\lim_{(x,y)\to(0,0)} \frac{x^2+2y^2}{x}$

Both of them are treated by using polar coordinate, compute $\lim_{r\to 0} f(r\cos\theta, r\sin\theta)$ and see if has a limit, and if so, if the limit depends on θ. If it does, this means that the value as $\lim_{(x,y)\to(0,0)}$ depends on which angle you approach, this $\lim_{(x,y)\to(0,0)} f(x,y)$ wouldn't exist. If the limit for $r \to 0$ doesn't depend on θ (as long as θ is taken so that $(r\cos\theta, r\sin\theta)$ is in the domain of f), then you may say that limit gives the limit $\lim_{(x,y)\to(0,0)} f(x,y)$.

(a): $\lim_{r\to 0} \frac{r\cos\theta}{\sqrt{r^2\cos^2\theta+2r^2\sin^2\theta}} = \lim_{r\to 0} \frac{\cos\theta}{\sqrt{\cos^2\theta+2\sin^2\theta}} = \frac{\cos\theta}{\sqrt{\cos^2\theta+2\sin^2\theta}}$. The right most term depends on θ so this means $\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist. If you got this limit $r \to 0$ right but didn't know what to conclude, you got 5pts. Note that r^2 inside the square root would come out as a single r so that cancels out with r on the top.
(b): $\lim_{r\to 0} \frac{r^2\cos^2\theta+2r^2\sin^2\theta}{r\cos\theta} = \lim_{r\to 0} r(\cos^2\theta + 2\sin^2\theta)/\cos\theta = 0$. One student was very careful to notice that this wouldn't work if $\cos\theta = 0$. That's right, but then for that, $x = 0$ so it's out of the domain of f, thus we don't care.

6 Let $f(x,y) = \sqrt{x^2+y^2}$.
(a)[10] Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$
(b)[5] Is f differentiable at each point $(0,0)$, $(2,2)$? Justify your answer.
(c) [10] Find the equation of the tangent plane of the graph for $z = f(x,y)$ at each point $(0,0,f(0,0))$ and $(2,2,f(2,2))$ if it exists.
(a) straightforward chain rule. If you applied chain rule in a wrong way, you got 6pt. If you did something more ridiculous than that, you got nothing. I'm pretty disappointed to see a lot of you making mistake in highschool algebra, pre-chain-rule calculus, etc. I hate to see things like $sqrtx^2 + y^2 = x + y$, or you take derivative of inside $\sqrt{*}$, leaving the result inside $\sqrt{*}$.
(b): There are two things to look at: whether 1: $\partial f/\partial x$, $\partial f/\partial y$ are defined at the given point, AND 2: if they are continuous around that point. The answer is that it's not differentiable at $(0,0)$ because $\partial f/\partial x$, $\partial f/\partial y$ are not defined there. f is differentiable at $(2,2)$ because $\partial f/\partial x$, $\partial f/\partial y$ are defined there and continuous around there.
If you only talked about 1, you got 4pt. In fact $\partial f/\partial x(0,0)$ thing is a bit tricky, so I didn't mind whatever result you got in this. In fact it is not defined, by a serious limit argument. And for 2, if you just said that they (partials) are conti at $(2,2)$, that's enough. But on the other hand, if you thought like "f must be differentiable because it's continuous", etc, you got zero.
(c): The answer would be, the tangent plane doesn't exist at $(0,0,f(0,0))$ because f is not differentiable. It does exist at $(2,2,f(2,2))$: just plug in the formula for the rest: everyone knew it. If you answered in (b) that f is not differentiable at $(0,0)$ but fabricated a tangent plane, you lost 3pts. If you said that there is no tangent plane at$(0,0)$ for an wrong reason, you lost 2pts.

This exam was quite easy as you can see from the average score. You do not need to be smart to do this well, you just need to understand. If you didn't do well, you should accept that you didn't understand something basic, not that you couldn't do frontflip.