Math 172 - HW 1
due April 7

March 31, 2015

1. We will need work pretty closely with complex numbers, so it is important that you get comfortable with them! Read over the Appendix A.2 in the text and do exercises 5,6, and 7 from Section A.2. (You are also encouraged, but not required to try your hand at the other exercises there.)

2. In class we showed that to solve an arbitrary cubic, it suffices to solve the corresponding reduced cubic equation $y^3 + py + q = 0$. We then used the change of variables $y = z - \frac{p}{3z}$ to obtain a degree 6 equation that is in fact quadratic in z^3, namely:

$$(z^3)^2 + qz^3 - \frac{p^3}{27} = 0.$$

   By the quadratic equation we found:

$$z^3 = \frac{1}{2}\left(-q \pm \sqrt{q^2 + \frac{4p^3}{27}}\right)$$

   We then *chose* a cube root z_1 of $\frac{1}{2}\left(-q + \sqrt{q^2 + \frac{4p^3}{27}}\right)$. We assumed that $p \neq 0$, so in particular, $z_1 \neq 0$. Finally, we observed that if we define z_2 to be $-\frac{p}{3z_1}$, then z_2 is a cube root of $\frac{1}{2}\left(-q - \sqrt{q^2 + \frac{4p^3}{27}}\right)$. Let ω be a nontrivial cube root of unity. We concluded that the three roots of $y^3 + py + q = 0$ can be expressed as:

$$y_1 = z_1 + z_2,$$

$$y_2 = \omega z_1 + \omega^2 z_2,$$

$$y_3 = \omega^2 z_1 + \omega z_2.$$

   These are known as Cardan's formulas.

   (a) Show that Cardan's formulas also gives the right answer for the roots of $y^3+py+q = 0$ when $p = 0$.

(b) Consider the polynomial $x^3 + x - 2 = 0$. Note that 1 is a root! Use Cardan's formulas (carefully!) to derive the surprising formula:

$$1 = \sqrt[3]{1 + \frac{2}{3}\sqrt{\frac{7}{3}}} + \sqrt[3]{1 - \frac{2}{3}\sqrt{\frac{7}{3}}}$$

(c) Show that $1+\frac{2}{3}\sqrt{\frac{7}{3}} = \left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{7}{3}}\right)^3$. Then use this to explain the surprising formula.