Math 132 - Midterm Review

April 27, 2015

Let F be a field throughout.

1. Here are some definitions that you need to know.
   1. The discriminant of a cubic polynomial.
   2. The polynomial ring $F[x_1, \ldots, x_n]$.
   3. The field of rational functions $F(x_1, \ldots, x_n)$.
   4. The elementary symmetric polynomials in $F[x_1, \ldots, x_n]$.
   5. Symmetric polynomials in $F[x_1, \ldots, x_n]$.
   6. A field extension L of F.
   7. If $L \supset F$ is a field extension, what does means for an element $\alpha \in L$ to algebraic? transcendental?
   8. What is the minimal polynomial of an algebraic element?
   9. If $L \subset F$ is a field extension and $\alpha_1, \ldots, \alpha_n \in L$, what is the ring $F[\alpha_1, \ldots, \alpha_n]$? What is the field $F(\alpha_1, \ldots, \alpha_n)$?
   10. If $L \subset F$ is a field extension, how is its degree defined?
   11. A finite field extension.
2. Here are the main results/techniques that we have covered and you should know.
   (a) How to solve a cubic equation! (Cardan's formula)
   (b) The relationship between the discriminant of a real cubic $f \in \mathbb{R}[x]$ and the number of real roots of f. [Thm 1.3.1]
   (c) If F is a field, L is a ring containing F, and $\alpha_1, \ldots, \alpha_n \in L$, then there is a natural evaluation homomorphism $F[x_1, \ldots, x_n] \to L$, taking a polynomial f to $f(\alpha_1, \ldots, \alpha_n)$. [Thm 2.1.2]
   (d) The connection between elementary symmetric polynomials and the coefficients of a polynomial with roots $\alpha_1, \ldots, \alpha_n$. [Prop 2.1.4 and Cor 2.1.5]
   (e) The Fundamental Theorem of Symmetric Polynomials. (e.g., How to express a symmetric polynomial as a polynomial in the elementary symmetric polynomials.) [Thm. 2.2.2, Cor. 2.2.5]

(f) If $f \in F[x]$ is irreducible, then $F[x]/\langle f \rangle$ is a field extension of F, in which f has a root (namely the coset $x + \langle f \rangle$. [Prop. 3.1.3]

(g) For any $f \in F[x]$, there exists a field extension of F containing all the roots of f. [Thm. 3.1.4]

(h) The Fundamental Theorem of Algebra. [Thm. 3.2.4]

(i) If $\alpha \in L$ is algebraic over F and $p \in F[x]$ the minimal polynomial of α, then $F[\alpha]$ is isomorphic to the quotient $F[x]/\langle p \rangle$. [Lem. 4.1.13]

(j) If $\alpha \in L$ is algebraic over F, then $F(\alpha) = F[\alpha]$ (and more generally $F(\alpha_1, \dots, \alpha_n) = F[\alpha_1, \dots, \alpha_n]$). [Prop 4.1.14 and 4.1.15]

(k) An element α in a field extension L of F is algebraic if and only if the degree $[F(\alpha) : F]$ is finite. In that case, $[F(\alpha) : F]$ is equal to the degree of the minimal polynomial of α and $\{1, \alpha, \dots, \alpha^{n-1}\}$ forms a basis for $F(\alpha)$ as a vector space over F. [Prop. 4.3.4]

(l) The Tower Theorem [Thm. 4.3.8]: if we have a "tower" of field extensions $F \subset K \subset L$, then: $[L : F] = [L : K][K : F]$.

(m) The Schönemann-Eisenstein criterion. [Thm 4.2.3] (To be covered on Tuesday April 28th)