HW 2
due April 20

1. The goal of this question is to prove that if $\phi \in F(x_1, \dots, x_n)$ is symmetric, then ϕ is a rational function in $\sigma_1, \dots, \sigma_n$ with coefficients in F. To start, we know that $\phi = A/B$, where A and B are in $F[x_1, \dots, x_n]$. Note that A and B need not be symmetric - we are only assuming that the quotient is!

   To help us along, consider the polynomial

   $$C = \prod_{\sigma \in S_n \setminus \{e\}} \sigma \cdot B.$$

   (a) Show that BC is a symmetric polynomial.

   (b) Show that AC is also a symmetric polynomial.

   (c) Use the previous parts to show that ϕ is indeed a rational function in the elementary symmetric polynomials with coefficients in F.

2. Let the roots of $y^3 + y^2 - 3y + 5$ be $\alpha, \beta, \gamma \in \mathbb{C}$. Find polynomials with integer coefficients with the following roots:

   (a) $\alpha\beta, \alpha\gamma$, and $\beta\gamma$.

   (b) $\alpha + 1, \beta + 1$, and $\gamma + 1$.

   (c) α^2, β^2, and γ^2.

3. Let $f = x^3 - x + 1 \in \mathbb{F}_3[x]$. Show that f is irreducible over $\mathbb{F}_3$. Let L be the splitting field of f over $\mathbb{F}_3$. Compute the degree $[L : \mathbb{F}_3]$ and the number of elements in L.

4. Determine the splitting field and its degree over $\mathbb{Q}$ for $x^4 + x^2 + 1$.