Math 131 Review Problems

January 7, 2015

Recall the definitions of: subspace, direct sum, span, linear (in)dependence, basis, dimension, linear transformation, kernel and image.

1. True/False: Determine which of the following statements are true. For each true statement, give a proof and for each false statement produce a counterexample.
   (a) If V is a vector space and W is a subset of V that is a vector space, then W is a subspace of V.
   (b) The empty set is a subspace of every vector space.
   (c) If V is a vector space other than the zero vector space, then V contains a subspace W such that $W \neq V$.
   (d) The intersection of any two subsets of V is a subspace of V.
   (e) An $n \times n$ diagonal matrix can never have more than n nonzero entries.

If S_1 and S_2 are non-empty subsets of a vector space V, the **sum** of S_1 and S_2, denoted $S_1 + S_2$, is the set $\{x + y | x \in S_1, y \in S_2\}$.

2. Let W_1 and W_2 be subspaces of a vector space V.
   (a) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.
   (b) Prove that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

3. True/False: Determine which of the following statements are true. For each true statement, give a proof and for each false statement produce a counterexample.
   (a) If S is a linearly dependent set, then each element of S is a linear combination of other elements of S.
   (b) Any set containing the zero vector is linearly dependent.
   (c) The empty set is linearly dependent.
   (d) Subsets of linearly dependent sets are linearly dependent.
   (e) Subsets of linearly independent sets are linearly independent.

(f) If $a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$ and $x_1, x_2, \ldots, x_n$ are linearly independent, then all the scalars a_i are zero.

4. True/False: Determine which of the following statements are true. For each true statement, give a proof and for each false statement produce a counterexample.
   (a) Every vector space that is generated by a finite set has a basis.
   (b) A vector space cannot have more than one basis.
   (c) The dimension of $\mathrm{Mat}_{m\times n}(\mathbb{F})$ is $m+n$.

5. Let W_1 and W_2 be subspaces of a vector space V having dimensions m and n, respectively, where $m \geq n$.
   (a) Prove that $\dim(W_1 \cap W_2) \leq n$.
   (b) Prove that $\dim(W_1 + W_2) \leq m + n$.

6. Show that the kernel and image of a linear transformation $T : V \to W$ are linear subspaces of V and W, respectively.

7. True/False: Determine which of the following statements are true. For each true statement, give a proof and for each false statement produce a counterexample. In what follows, V and W are finite-dimensional vector spaces (over $\mathbb{F}$) and T is a function from V to W.
   (a) If T is linear, then T preserves sums and scalar products.
   (b) If $T(x+y) = T(x) + T(y)$, then T is linear.
   (c) T is one-to-one if and only if $\{v \in V | T(v) = 0\} = \{0\}$.
   (d) if T is linear, then $T(0_V) = 0_W$.
   (e) If T is linear, then $\dim\ker(T) + \dim\mathrm{im}(T) = \dim(W)$.
   (f) If T is linear, then T carries linearly independent subsets of V onto linearly independent subset of W.
   (g) If $T, U : V \to W$ are both linear and agree on a basis of V, then $T = U$.
   (h) Given $x_1, x_2 \in V$ and $y_1, y_2 \in W$, there exists a linear transformation $T : V \to W$ such that $T(x_1) = y_1$ and $T(x_2) = y_2$.