Math 132 - Final Review

March 10, 2015

1. As well as those from the midterm review sheet, you should also know the definitions of the following concepts:

    1. An **inner product space** on a real or complex vector space.
    2. The **norm** (or length) of a vector in an inner product space.
    3. What does it mean for a set of vectors to be **orthogonal**? **orthonormal**?
    4. The **Gram-Schmidt procedure**.
    5. The **orthogonal subspace** $S^\perp$ for any subset $S \subset V$ of an inner product space.
    6. The **adjoint** $T^* : W \to V$ of a linear map $T : V \to W$ between inner product spaces V and W.
    7. The **least squares approximation** of a data set.
    8. A **normal operator** on an inner product space.
    9. A **self-adjoint operator** on an inner product space.
    10. A **unitary operator** on a complex inner product space.
    11. A **orthogonal operator** on a real inner product space.

2. Here are the main results that we have covered since the midterm. You also need to know the ones on the midterm review sheet. Assume throughout that V is an inner product space and that T is a linear operator on V.

    (a) The Cauchy-Schwarz Inequality

    (b) The Triangle Inequality

    (c) **Orthogonal projection**: Let $W \subset V$ be a finite-dimensional subspace and $v \in V$:

    - There exist unique vectors $w \in W$ and $z \in W^\perp$ such that $v = w + z$.
    - If $\beta = \{v_1, \dots, v_k\}$ is an orthonormal basis of W, then $w = \sum_{i=1}^{k} \langle v, v_i \rangle v_i$.
    - The vector w is the unique vector in W 'closest' to v in the sense that for any $u \in W$, $||v - w|| \geq ||v - u||$ and equality holds if and only if $w = u$.

    (d) If V is finite-dimensional and W is a subspace of V, then

$$\dim(V) = \dim(W) + \dim(W^\perp).$$

(e) Let V be a finite-dimensional over a field $\mathbb{F}$. Let $g : V \to \mathbb{F}$ be a linear map. Then there exists a unique vector $y \in V$ such that $g(x) = \langle x, y\rangle$ for all $x \in V$.

(f) The adjoint of a linear map is a well-defined linear map.

(g) Suppose V is finite-dimensional and β is an *orthonormal* basis for V. Then if T is a linear operator on V,

$$[T^*]_\beta = \overline{[T]^t_\beta}.$$

(h) Assume V is finite-dimensional. If T has an eigenvector, then so does T^*. (Warning: This does not mean that an eigenvector of T is an eigenvector of T^*!)

(i) Schur's Theorem: Assume V is finite-dimensional and that the characteristic polynomial of T splits (e.g., if V is complex). Then there exists an orthonormal basis $\beta = \{v_1, \dots, v_n\}$ for V such that $[T]_\beta$ is upper-triangular.

(j) If T is a normal operator, then

- $||T(x)|| = ||T^*(x)||$, for all $x \in V$.
- If x is an eigenvector of T, then x is also an eigenvector of T^*. In fact, if $T(x) = \lambda x$, then $T^*(x) = \overline{\lambda} x$.
- If x_1, x_2 are eigenvectors with distinct eigenvalues, then $\langle x_1, x_2\rangle = 0$.

(k) Assume that V is a finite-dimensional *complex* inner product space. Then T is normal if and only if there exists an orthonormal basis for V consisting of eigenvectors of T.

(l) Assume that T is self-adjoint. Then:

- The eigenvalues of T are real,
- The characteristic polynomial of T splits.

(m) Assume that V is a finite-dimensional *real* inner product space. Then T is self-adjoint if and only if there exists an orthonormal basis for V consisting of eigenvectors of T.

(n) Suppose that V is finite-dimensional. Then the following are equivalent:

- $T^*T = I = TT^*$,
- $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for any $x, y \in V$.
- for any orthonormal basis β for V, $T(\beta)$ is orthonormal,
- there exists an orthonormal basis β for V such that $T(\beta)$ is orthonormal,
- $||T(x)|| = ||x||$ for any $x \in V$.

(o) Assume that V is a finite-dimensional *real* inner product space. T is both self-adjoint and orthogonal if and only if there exists an orthonormal basis β of eigenvectors of T with eigenvalues equal to 1 or -1.

(p) Assume that V is a finite-dimensional *complex* inner product space. T is unitary if and only if there exists an orthonormal basis β of eigenvectors of T with eigenvalues of absolute value 1.