Math 132 - HW 11
due February 25

February 18, 2015

1. Let V and W be finite-dimensional vector spaces over $\mathbb{F}$. Let $\mathcal{L}(V, W)$ denote the vector space of linear transformations from $V \to W$. Let $\beta = \{v_1, \ldots, v_n\}$ be a basis for V and $\gamma = \{w_1, \ldots, w_m\}$ a basis for W. Given a linear transformation $T : V \to W$, recall that $[T]_\beta^\gamma$ denotes the $n \times m$-matrix with ij-th entry given by $a_{ij} \in \mathbb{F}$, the coefficient of w_i in $T(v_j)$:
$$T(v_j) = a_{1j}w_1 + a_{2j}w_2 + \ldots + a_{mj}w_m.$$
   (a) Show that the function $\mathcal{L}(V, W) \to \mathrm{Mat}_{n,m}(\mathbb{F})$ defined by $T \mapsto [T]_\beta^\gamma$ is a linear map.

   (b) Show that the linear map defined in the previous part is an isomorphism of vector spaces.

   (c) What is the dimension of $\mathrm{Mat}_{n,m}(\mathbb{F})$?

   (d) Use the previous parts to compute the dimension of $\mathcal{L}(V, W)$.

   (e) Recall that when $\mathbb{F} = \mathbb{R}$, the vector space $\mathcal{L}(V, \mathbb{R})$ is the same as the dual vector space V^* that we defined on the worksheet. In class we proved that if V is an inner product space over $\mathbb{R}$, then there is an injective linear map $V \to V^*$ given by $y \mapsto g_y$, where $g_y : V \to \mathbb{R}$ is given by $g_y(x) = \langle x, y \rangle$. Use the computation from the previous part to conclude that the map $V \to V^*$ is an isomorphism.

2. Let T be a linear operator on an inner product space V, with adjoint T^*. Let $A = T + T^*$ and $B = TT^*$. Prove that $A^* = A$ and $B^* = B$.

3. Let V be a finite-dimensional inner product space, and let T be a linear operator on V. Prove that if T is invertible, then the adjoint T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.