Math 132 - HW 6
due January 28

January 21, 2015

1. True/False: Determine which of the following statements are true. For each true statement, give a proof and for each false statement produce a counterexample.
   (a) There exists a linear operator T with no T-invariant subspace.
   (b) Let T be a linear operator on a finite-dimensional vector space V, and let v and w be elements of V. If W is the T-cyclic subspace generated by v, W' is the T-cyclic subspace generated by w, and $W = W'$, then $v = w$.
   (c) If T is a linear operator on a finite-dimensional vector space V, then for any $v \in V$ the T-cyclic subspace generated by v is the same as the T-cyclic subspace generated by $T(v)$.

2. Let T be a linear operator on a vector space V, and let W be a T-invariant subspace of V. Prove that W is $g(T)$-invariant for any polynomial $g(t) \in \mathbb{F}[t]$.

3. Let T be a linear operator on a vector space V, let v be a nonzero element of V, and let W be the T-cyclic subspace of V generated by v. For any $w \in V$, prove that $w \in W$ if and only if there exists a polynomial $g(t)$ such that $w = g(T)(v)$.

4. Let A be the $k \times k$-matrix:
$$\begin{pmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & & 0 & -a_1 \\ 0 & 1 & & 0 & -a_1 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 0 & -a_{k-2} \\ 0 & 0 & \dots & 1 & -a_{k-1} \end{pmatrix}$$
where $a_0, a_1, \dots, a_{k-1}$ are arbitrary scalars. Prove that the characteristic polynomial of A is
$$(-1)^k(a_0 + a_1 t + \dots + a_{k-1}t^{k-1} + t^k).$$
Hint: Use induction on k.