Math 132 - Midterm Review

January 28, 2015

1. Here are definitions that you need to know. Items (1-14) you should have learned (at least seen!) in Math 132. After each item I have indicated the dependencies in brackets. For example, item two ends with [1], meaning that one needs to know the definition of a vector space to really understand the definition of a subspace. (I probably missed some further dependencies, but hopefully the ones I did catch will help you identify what you need to learn if there are things you don't understand.)

    1. A vector space over a field $\mathbb{F}$.
    2. A subspace of a vector space. [1]
    3. A linear combination of a set of vectors in a vector space. [1]
    4. The span of a set of vectors in a vectors space. [1-3]
    5. A linearly independent/dependent set in a vector space. [1,3]
    6. A basis of a vector space. [1-5]
    7. The dimension of a vector space. [1-6]
    8. A linear transformation $f : V \to W$ between two vector spaces V and W. [1]
    9. A linear operator on a vector space V. [1,8]
    10. The kernel of a linear transformation. [1,2,8]
    11. The image of a linear transformation. [1,2,8]
    12. An isomorphism between two vector spaces. [1,2,8,10,11]
    13. An invertible linear transformation. [1,2,8,10,11,12]
    14. The matrix representation of a linear operator T on a finite dimensional vector space V in terms of an ordered basis $\beta = \{v_1, \ldots, v_n\}$. [1-6,8]
    15. An eigenvector of a linear operator $L : V \to V$. [1,8,9]
    16. An eigenvalue of a linear operator $L : V \to V$. [1,8,9]
    17. The characteristic polynomial of an $n \times n$ matrix A. [determinants/row reduction]
    18. The characteristic polynomial of a linear operator $L : V \to V$ on a finite dimensional vector space V. [1-6,8,9,14]
    19. What does it mean for two matrices to be similar?

20. A diagonalizable matrix. [19]
21. A diagonalizable linear operator on a finite dimensional vector space. [1-6,8,14-16,18,19]
22. The λ-eigenspace of a linear operator $L : V \to V$, where λ is an eigenvalue of L. [1,2,8,9,10]
23. The multiplicity of a root of a polynomial.
24. The multiplicity of an eigenvector of a linear transformation. [1-6,8,9,14,18,23]
25. What does it mean for a polynomial $f(t) \in \mathbb{F}$ to split over $\mathbb{F}$?
26. An L-invariant subspace of a vector space V, where $L : V \to V$ is a linear operator on V. [1,2,8,9]
27. The restriction of a linear operator $L : V \to V$ to an L-invariant subspace. [1,2,8,9,26]
28. The L-cyclic subspace generated by a vector $v \in V$, where L is a linear operator on a vector space V. [1,2,8,9,26]
29. Sum of subspaces and direct sum [1,2]
30. If $f(t) \in \mathbb{F}[t]$ is a polynomial and $T : V \to V$ is a linear operator, what does $f(T)$ mean? [1,8,9]

2. Here are the main results that we have covered and you should know.
   (a) If T is a linear operator on a vector space V and $v_1, \dots, v_k$ are eigenvectors with corresponding eigenvalues that are distinct, then $\{v_1, \dots, v_k\}$ is linearly independent.
   (b) The characteristic polynomial of any diagonalizable linear operator splits. (But if the characteristic polynomial splits it doesn't mean that the linear operator is is diagonalizable!)
   (c) Let T be a linear operator on a finite-dimensional vector space V, and let λ be an eigenvalue of T with multiplicity m. Then $\dim(E_\lambda) \leq m$.
   (d) Let T be a linear operator on a finite dimensional vector space V, then T is diagonalizable if and only if both of the following conditions hold:
      - The characteristic polynomial of T splits, and
      - For each eigenvalue λ of T, the dimension of the λ-eigenspace $\dim(E_\lambda)$ is equal to the multiplicity of λ.
   (e) Let T be a linear operator on a finite-dimensional vector space V, and let W be a T-invariant subspace of V. Then the characteristic polynomial of $T|_W$ divides the characteristic polynomial of T.
   (f) (The Cayley-Hamilton Theorem) Let T be a linear operator on a finite-dimensional vector space V over a field $\mathbb{F}$, and let $\chi_T(t) \in \mathbb{F}[t]$ denote the characteristic polynomial of T. Then $\chi_T(T) = 0$, where the 0 on the right hand side denotes the zero transformation. In other words, T "satisfies" its characteristic polynomial!