Math 132 - Extra credit quiz 1

1. Let V be a vector space. Suppose U and W are subspaces of V. Using set notation, give the definition of the subspace $U+W$.

$$U + W := \{u + w | u \in U, w \in W\}$$

2. Suppose $T \in \mathcal{L}(V)$ and $T^2 = T$. Prove that $V = \text{null } T + \text{range } T$.

   This problem was a part of your homework. Here are two possible solutions. (The second requires that V be finite dimensional)

   Proof 1: We need to show that any $v \in V$ can be expressed as a sum of an element of null T and an element of range T. Claim: $v = (v - T(v)) + T(v)$ does the job.

   By definition $T(v) \in \text{range } T$, so it suffices to check that $v - T(v) \in \text{null } T$.

   Applying the linear operator T to the vector $v - Tv$, we find:

$$T(v - Tv) = Tv - T^2v = Tv - Tv = 0.$$

   Thus, $v - Tv \in \text{null } T$ and we are done.

   Proof 2: As $\text{null } T + \text{range } T \subset V$ is a subspace (and assuming V is finite dimensional),

$$\dim(\text{null } T + \text{range } T) \leq \dim V$$

   with equality if and only if $\text{null } T + \text{range } T = V$. Using the dimension of a sum formula

$$\dim(\text{null } T + \text{range } T) = \dim(\text{null } T) + \dim(\text{range } T) - \dim(\text{null } T \cap \text{range } T),$$

   and by the fundamental theorem of linear maps (aka rank-nullity) this is equal to:

$$= \dim V - \dim(\text{null } T \cap \text{range } T).$$

   Thus it suffices to prove that $\text{null } T \cap \text{range } T = 0$.

   Suppose $v \in \text{null } T \cap \text{range } T$. Then $v = Tw$ for some $w \in V$ and $Tv = 0$. But then $v = Tw = T(Tw) = Tv = 0$.

3. Prove or give a counterexample: If $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$ are linearly independent lists of vectors in a vector space V, then $v_1 + w_1, \ldots, v_m + w_m$ is linearly independent.

   False. For a simple counterexample, consider a nonzero element v of a vector space V (For example, $1 \in \mathbb{R}$). Note that $v_1 = v$ is a linearly independent list and so is $w_1 = -v$. The sum $v_1 + w_1 = v + (-v) = 0$, however, is not a linearly independent list.