Math 132 - Extra credit quiz 2

1. Let $U = \{(a,b,c) \in \mathbb{R}^3 | 2a - b = 0, a + b + c = 0\} \subset \mathbb{R}^3$. Endowing $\mathbb{R}^3$ with the standard dot product, express the vector $(1,0,0)$ as a sum $u + w$, where $u \in U$ and $w \in U^\perp$.

   Solution: Let $v := (1,0,0)$. By definition, $u = P_U(v)$, the orthogonal projection of v onto U (and so $w = v - u$). To compute $P_U(v)$, we first find an orthonormal basis of U.

   Note that $U = \{(a,b,c) \in \mathbb{R}^3 | 2a - b = 0, a + b + c = 0\} = \{(a, 2a, c) | a + 2a + c = 0\} = \{(a, 2a, -3a) | a \in \mathbb{R}\} = \operatorname{span}((1,2,-3))$. Thus, the single vector $(1,2,-3)$ forms a basis of U.

   To obtain an orthonormal basis of U, we apply the Gram-Schmidt procedure. In this case, because our basis has only one element, we just have to normalize. Doing so, we obtain $e_1 = (1,2,-3)/||(1,2,-3)|| = \frac{1}{\sqrt{14}}(1,2,-3)$, which is an orthonormal basis of U.

   Thus $u = P_U(v) = \langle v, e_1 \rangle e_1 = \langle (1,0,0), \frac{1}{\sqrt{14}}(1,2,-3) \rangle \frac{1}{\sqrt{14}}(1,2,-3) = \frac{1}{14}(1,2,-3)$, and $w = v - u = (1,0,0) - \frac{1}{14}(1,2,-3) = \frac{1}{14}(13,-2,3)$.

2. Let U be as above. What is the dimension of U and why?

   Solution: As we showed in (1.), U has a basis of length one. Thus $\dim(U) = 1$.

3. Suppose that W and W' are subspaces of a vector space V, such that $\dim(W) = \dim(W') = 3$ and $\dim(V) = 5$. Prove or give a counterexample: $W \cap W' \neq \{0\}$.

   True! Recall that $W + W'$ is a subspace of V and so $\dim(W + W') \leq \dim(V)$. On the other hand, we have a formula for the dimension of a sum of subspaces:

   $$\dim(W + W') = \dim(W) + \dim(W') - \dim(W \cap W')$$

   Putting these two facts together, we have:

   $$\dim(V) \geq \dim(W) + \dim(W') - \dim(W \cap W'),$$

   Thus, $5 \geq 3 + 3 - \dim(W \cap W')$ or in other words $\dim(W \cap W') \geq 1$. We conclude that $\dim(W \cap W') \neq 0$ and hence $W \cap W' \neq \{0\}$.