Math 132 - Extra credit quiz 3

1. Let V and W be vectors spaces and $T : V \to W$ be a linear map. Show that if T is a injective and $v_1, \ldots, v_k$ is a linearly independent list of vectors in V, then $Tv_1, \ldots, Tv_k$ is a linearly independent list in W.

   Proof: Suppose $a_1, \ldots, a_k \in \mathbb{F}$ and that

   $$a_1Tv_1 + \cdots + a_kTv_k = 0.$$

   To show that $Tv_1, \ldots, Tv_k$ is a linearly independent list, we must show that this implies $a_1 = \cdots = a_k = 0$.

   By the linearity of T:

   $$T(a_1v_1 + \cdots + a_kTv_k) = 0.$$

   Thus, $a_1v_1 + \cdots + a_kTv_k \in \text{null}(T)$. As T is injective, this implies:

   $$a_1v_1 + \cdots + a_kTv_k = 0.$$

   But $v_1, \ldots, v_k$ is linearly independent, so we can conclude that $a_1 = \cdots = a_k = 0$, as was to be shown.

2. Let $T \in \mathcal{L}(\mathbb{C}^3)$. Suppose that T has eigenvectors $(1, 1, 0), (1, -1, 0), (0, 0, i)$ with respective eigenvalues $2, 4 + i$ and -1.

   (a) Is T normal? Why or why not?

   Yes! After normalizing the given eigenvectors, we obtain an orthonormal basis of $\mathbb{C}^3$ consisting of eigenvectors for T. The complex spectral theorem tells us that this implies that T is normal.

   (b) Is T self-adjoint? Why or why not?

   No. Self-adjoint operators have only real eigenvalues.

   Note: A common mistake was to consider a matrix with columns (or rows) given by the eigenvectors. Note that such a matrix does not represent that operator T (e.g., $(1, 1, 0)$ is not an eigenvector of such a matrix).