HOMEWORK PROBLEMS

Due Tuesday Feb. 5th -
1.0.7 (a) Let R be a normal domain with field of fractions K and let $S \subset R$ be a multiplicative subset. Prove that the localization R_S is normal.
(b) Let R_α, $\alpha \in A$ be normal domains with the same field of fractions K. Prove that the intersection $\cap_{\alpha \in A} R_\alpha$ is normal.

1.1.11 Let T_N be a torus with character lattice M. Then every point $t \in T_N$ gives an evaluation map $\phi_t : M \to \mathbb{C}^*$ defined by $\phi_t(m) = \chi^m(t)$. Prove that ϕ_t is a group homomorphism and that the map $t \mapsto \phi_t$ induces a group isomorphism

$$T_N \cong \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*).$$

Due Thursday Feb. 7th -
1.1.6 Different sets of lattice points can parametrize the same affine toric variety, even though these parametrizations behave slightly differently. In this exercise you will consider the parametrizations

$$\Phi_1(s,t) = (s^2, st, st^3) \qquad \text{and} \qquad \Phi_2(s,t) = (s^3, st, t^3).$$

(a) Prove that Φ_1 and Φ_2 both give the affine toric variety $Y = \mathbb{V}(xz - y^3) \subset \mathbb{C}^3$.
(b) We can regard Φ_1 and Φ_2 as maps

$$\Phi_1 : \mathbb{C}^2 \to Y \qquad \text{and} \qquad \Phi_2 : \mathbb{C}^2 \to Y.$$

Prove that Φ_2 is surjective and that Φ_1 is not. The images are called *toric sets* - the text gives references to papers studying when a toric set equals the corresponding affine toric variety.

1.1.12 Consider tori T_1 and T_2 with character lattices M_1 and M_2. As we saw in class (Example 1.1.13 in the book), the coordinate rings of T_1 and T_2 are $\mathbb{C}[M_1]$ and $\mathbb{C}[M_2]$. Let $\Phi : T_1\ T_2$ be a morphism that is a group homomorphism. Then Φ induces maps

$$\hat{\Phi} : M_2 \to M_1 \qquad \text{and} \qquad \Phi^* : \mathbb{C}[M_2] \to \mathbb{C}[M_1]$$

by composition. Prove that Φ^* is the map of semigroup algebras induced by the map $\hat{\Phi}$ of affine semigroups.

Here is another problem to think about. You do not need to write it up:
1.1.10 Prove that $I = \langle x^2 - 1, xy - 1, yz - 1 \rangle$ is the lattice ideal for the lattice

$$L = \{(a,b,c) \in \mathbb{Z}^3 | a + b + c \equiv 0 \mod 2\} \subset \mathbb{Z}^3.$$

Also compute the primary decomposition of I to show that I is not prime.

Due Tuesday Feb. 12th -
1.2.1 Let τ be a face of a polyhedral cone σ. If $v, w \in \sigma$ and $v + w \in \tau$, show that $v, w \in \tau$. Hint: Write $\tau = H_m \cap$ for $m \in \sigma^\vee$.

1.2.13 Consider the cone $\sigma = \mathrm{Cone}(3e_1 - 2e_2, e_2) \subset \mathbb{R}^2$.
(a) Describe $\sigma^\vee$ and find generators of $\sigma^\vee \cap \mathbb{Z}^2$. Draw both cones together with facet normals for each edge.
(b) Compute the toric ideal of the affine toric variety U_σ. Note the connection to problem 1.1.6 from the last problem set.

Here is another problem to think about. You do not need to write it up:
1.2.2 Let $\sigma \subset N_\mathbb{R}$ be a cone.
(a) Show that if $u \in \sigma$, then $u \in \mathrm{Relint}(\sigma)$ if and only if $\langle m, n\rangle > 0$ for all $m \in \sigma^\vee - \sigma^\perp$ if and only if $\sigma^\vee \cap u^\perp = \sigma^\perp$.
(b) Let $\tau \preceq \sigma$ and fix $m \in \sigma^\vee$. Prove that

$$m \in \tau^* \iff \tau \subset H_m \cap \sigma,$$
$$m \in \mathrm{Relint}(\tau^*) \iff \tau = H_m \cap \sigma.$$

Due Thur. Feb. 14th -
1.3.7 Let p be a point of an irreducible affine variety V. Then p gives the maximal ideal $\mathfrak{m} = \{f \in \mathbb{C}[V] | f(p) = 0\}$ as well as the maximal ideal $\mathfrak{m}_{V,p} \subset \mathcal{O}_{V,p}$. Prove that the natural map $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}_{V,p}/\mathfrak{m}_{V,p}^2$ is an isomorphism of $\mathbb{C}$-vector spaces.

Due Thur. Feb. 19th -
1.3.11 Let $d > 1$ be an integer and $\mu_d = \{\zeta \in \mathbb{C}^* | \zeta^d = 1\}$. Consider the lattices:

$$N' = \mathbb{Z}^2 \subset N = \{(a/d, b/d) | a, b \in \mathbb{Z}, a/d - b/d \in \mathbb{Z}\},$$

and the cone $\sigma = \mathrm{Cone}(e_1, e_2) \subset N'_\mathbb{R} = N_\mathbb{R}$.
Show that $U_{\sigma,N} = \hat{C}_d = \mathbb{C}^2/\mu_d = U_{\sigma,N'}/\mu_d$ where μ_d acts on $\mathbb{C}^2$ by $\zeta \cdot (x, y) = (\zeta x, \zeta y)$.
