HOMEWORK PROBLEMS II

Due Thursday Feb. 21st -
2.1.4 Let $\mathcal{A} = \{e_1, e_2, e_1 + 2e_2, 2e_1 + e_2\} \subset \mathbb{Z}^2$. Compute $\mathbb{Z}\mathcal{A}$ and $\mathbb{Z}'\mathcal{A}$ and show that $\mathbb{Z}'\mathcal{A}$ has index two in $\mathbb{Z}\mathcal{A}$. Conclude that $Y_{\mathcal{A}} \neq \hat{X_{\mathcal{A}}}$ and that the map of tori $T_{Y_{\mathcal{A}}} \to T_{X_{\mathcal{A}}}$ is two-to-one. Lastly, check that $\mathbb{I}(Y_{\mathcal{A}})$ is not homogeneous.

2.1.8 Let $d \geq 2$ be an integer. Consider a set $\mathcal{A} = \{(a_0, b_0), \ldots, (a_n, b_n)\}$ where the $a_i \in \mathbb{N}$ form a decreasing sequence and $a_i + b_i = d$ for every i. This give a projective cure $X_{\mathcal{A}} \subset \mathbb{P}^n$. We assume that $n \geq 2$.

(a) If $a_0 < d$ or $a_n > 0$, explain how to obtain the same projective curve using monomials of strictly smaller degree.

(b) Assume $a_0 = d$ and $a_n = 0$. Use the cover whose elements are labelled by the vertices of $\mathrm{Conv}(\mathcal{A})$ to show that C is smooth if and only if $a_1 = d - 1$ and $a_{n-1} = 1$. (Hint: in one direction you might use the fact that smooth varieties are normal.)

Due Tuesday Feb. 26th -
2.2.13 Fix an integer $a \geq 1$ and consider the 3-simplex $P = \mathrm{Conv}(0, ae_1, ae_2, e_3)$.
(a) Describe the normal fan Σ_P.

(b) Show that P is normal and hence very ample.

2.3.6 (a) Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. Describe the normal fan Σ_Δ of the standard n-simplex $\Delta_n = \mathrm{Conv}(\{0, e_1, \ldots, e_n\})$. Draw pictures of the normal fan for $n = 2, 3$.

(b) For an integer $k \geq 1$, show that the toric variety $X_{k\Delta} \subset \mathbb{P}^{s_k - 1}$ is given by the Veronese embedding $\nu_k : \mathbb{P}^n \to \mathbb{P}^{s_k-1}$ defined using all monomials of total degree k is $\mathbb{C}[x_0, \ldots, x_n]$.

Eric asked for more challenging problems. Here are two:
Optional 1 Prove the theorem we stated in class that if P is a full dimensional lattice polytope of dimension $n \geq 2$, then kP is normal for all $k \geq n - 1$.
You might use the following sketch:
(1) Prove the statement when P is a simplex and has no interior lattice points.
(2) Show that every polytope is a union of lattice simplices.
(2) Show that every lattice simplex is a finite union of lattice simplices with no interior points.

Optional 2 Find a lattice polytope that is very ample, but not normal.

Due Thursday Feb. 28th -

3.1.5 Let $N = \mathbb{Z}^n$, $e_1, \dots, e_n \in N$ be the standard basis, and $e_0 = e_1 + \cdots + e_n$. Let Σ be the set of cones generated by all subsets of $\{e_0, \dots, e_n\}$ not containing $\{e_1, \dots, e_n\}$.
(a) Show that Σ is a fan in $N_{\mathbb{R}}$.
(b) Construct the affine open subsets covering the corresponding toric variety X_Σ, and give the gluing isomorphisms.
(c) Show that X_Σ is isomorphic to the blowup of $\mathbb{C}^n$ at the origin. Recall that the blowup is the subvariety of $\mathbb{P}^{n-1} \times \mathbb{C}^n$ given by $W = \mathbb{V}(x_i y_j - x_j y_i | 1 \leq i < j \leq n)$ where the x_i are homogeneous coordinates for $\mathbb{P}^{n-1}$ and the y_i coordinates for $\mathbb{C}^n$. Note that W is covered by affine open subsets $W_i = W_{x_i}$. Thus you should try to match these up with the affine open covering from (b).

Due Tuesday March 5th -
3.2.11 Recall the theorem of Sumihiro: If a torus T_N acts on a normal separated variety X, then every point of X has T_N-invariant affine open neighborhood. Using this result, we now have all the tools we need to show that for every normal separated toric variety X with torus T_N, there exists a fan Σ in $N_{\mathbb{R}}$ such that $X \cong X_\Sigma$.

By Sumihiro's Theorem X is covered by affine toric varieties U_σ for some collection of strongly convex rational polyhedral cones $\sigma \subset N_{\mathbb{R}}$. Let Σ be the set of all such cones and their faces.

(a) Check that for any $\sigma, \sigma' \in \Sigma$, $U_\sigma \cap U_{\sigma'}$ is an affine toric variety and thus corresponds to a cone τ in $N_{\mathbb{R}}$.

(b) Show that if $U_\sigma \cap U_{\sigma'} = U_\tau$, then $\tau = \sigma \cap \sigma'$.

(c) Show that if $\tau = \sigma \cap \sigma'$ for $\sigma, \sigma' \in \Sigma$, then τ is a face of both.

(d) Conclude that $X \cong X_\Sigma$.

Here are some hints if you want them:
(a) To show it is affine, use that X is separated.
(b) Recall that for a cone $\tau \subset N_{\mathbb{R}}$ and $u \in N$, $u \in \tau$ if and only if the limit $\lim_{z \to 0} \lambda^u(z)$ exists in U_τ.
(c) It suffices to show that if $\tau \subset \sigma$ and the induced map of affine toric varieties $U_\tau \to U_\sigma$ is an immersion, then τ is a face of σ. To do so, use the limit property from the hint for (b) and the characterization of faces as convex subsets $\tau \subset \sigma$ such that if $x + y \in \tau$ then both $x, y \in \tau$.

Due Thursday March 7th -
1. As Levent asked about it... Consider the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$ defined by taking $(a_0, \dots, a_n, b_0, \dots, b_m)$ to the point:

$$(a_0 b_0, a_0 b_1, \dots, a_0 b_m, a_1 b_0, \dots, a_1 b_m, \dots, a_n b_0, \dots, a_n b_m).$$

Describe the corresponding fans and compatible map of cocharacter lattices. Draw a picture of the case of $n = m = 1$.

2. Let $N = \mathbb{Z}^2$ and σ be the cone $\mathrm{Cone}(e_1, e_1 + de_2)$ for $d > 0$. Note that U_σ is singular at the origin if $d > 1$. Find a toric resolution of U_σ and describe the fiber over the origin.

3. Suppose $0 \to N' \to N \xrightarrow{\overline{\phi}} N'' \to 0$ is a short exact sequence of lattices with respective fans $\Sigma', \Sigma, \Sigma''$ that are compatible with the maps.
Suppose moreover:

- There exists a subfan $\hat{\Sigma} \subset \Sigma$ in $N_\mathbb{R}$ that lifts Σ'', meaning that for any $\sigma'' \in \Sigma''$, there exists a unique $\hat{\sigma} \in \hat{\Sigma}$ such that $\overline{\phi}(\hat{\sigma} \cap N) = \sigma'' \cap N''$.
- The sum $\hat{\sigma} + \sigma'$ is in Σ for $\hat{\sigma} \in \hat{\Sigma}$, $\sigma' \in \Sigma'$ and every cone in Σ is of this form.

Prove that the sequence $X_{\Sigma'} \to X_\Sigma \to X_{\Sigma''}$ is a locally trivial fibration.

Due Tuesday March 12th -
1. Classify smooth complete toric varieties with divisor class group isomorphic to $\mathbb{Z}$. (Courtesy of Peter McNamara)

Due Thursday March 14th -
4.2.10 Let X_P be the toric variety of the octahedron $P = \mathrm{Conv}(\{\pm e_i\}) \subset \mathbb{R}^3$.

(a) Show that $\mathrm{Cl}(X_P) \cong \mathbb{Z}^5 \oplus (\mathbb{Z}/2)^2$.

(b) Show that $\mathrm{Pic}(X_P) \cong \mathbb{Z}$.

Due Tuesday March 26th -
Have a good break!
